Technical Report for ID TNET-2015-00294 "Optimal delay characteristic when the number of users is comparable to the number of rounds"

Simon Oya, Fernando Pérez-González, and Carmela Troncoso

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In this document, we explain the similarity of the solutions of the optimal pool mix problem and the quasi-optimal pool mix problem, presented in the main manuscript ID TNET-2015-00294. We do this under the assumptions that the number of users in the system N is comparable to the number of rounds observed ρ, while $\rho \to \infty$. Please refer to the original document for a description of the notation used here.

We recall that the optimal pool problem is the solution to

$$\begin{aligned} \mathbf{d}_{opt} = \underset{\mathbf{d}}{\text{argmax}} \quad & \text{Tr}\left\{\mathbf{M}\mathbf{C}_e\mathbf{M}\right\} \\ \text{subject to} \quad & \sum_{k=0}^{\rho-1} d_k = 1, \quad d_k \geq 0, \ \forall k \\ & \sum_{k=1}^{\rho-1} k \cdot d_k \leq \bar{\delta} \end{aligned} \tag{1}$$

where

$$\mathbf{C}_e = \text{E}\left\{(\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{D}^T\mathbf{\Sigma}_{\mathbf{Y}|\mathbf{X}}\mathbf{D}\mathbf{X}(\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1}\right\}, \tag{2}$$

and

$$\mathbf{\Sigma}_{\mathbf{Y}|\mathbf{X}} = \text{diag}\left\{\mathbf{D}\mathbf{X}\cdot\mathbf{1}_N\right\} - \mathbf{D}\cdot\text{diag}\left\{\mathbf{X}\cdot\mathbf{r}_1\right\}\cdot\mathbf{D}^T + \mathbf{D}\cdot\left[\sum_{i=1}^{N}\left(\mathbf{X}_i\mathbf{X}_i^T \circ \boldsymbol{E}_i\right)\cdot r_2(i)\right]\cdot\mathbf{D}^T. \tag{3}$$

Note that, as now N is comparable to ρ, we can no longer approximate $\frac{1}{\rho}\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}$ by its expected value $\mathbf{R}_{xx} = \frac{1}{\rho}\text{E}\left\{\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}\right\}$. Therefore, we cannot write $(\frac{1}{\rho}\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1} = \mathbf{R}_{xx}^{-1}$.

In the quasi-optimal pool problem, the objective function is a simplified version of the one above:

$$\begin{aligned} \mathbf{d}'_{opt} = \underset{\mathbf{d}}{\text{argmax}} \quad & \rho\cdot\text{Tr}\left\{\text{E}\left\{\mathbf{M}\cdot(\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1}\cdot\mathbf{M}\right\}\right\} \\ \text{subject to} \quad & \sum_{k=0}^{\rho-1} d_k = 1, \quad d_k \geq 0, \ \forall k \\ & \sum_{k=1}^{\rho-1} k \cdot d_k \leq \bar{\delta} \end{aligned} \tag{4}$$

One possible strategy to increase the value of the optimization function in (1) consists in making $\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}$ "almost singular", so that after inverting this matrix in (2), performing the matrix multiplications and computing the trace, the value obtained is large. Note that we can also follow this strategy to increase (4). In the following section, we explain why this idea can be seen as a filter design problem, where we want to compute the filter that removes a specific set of frequency bins. Then, in Section 2 we show empirically that the solution to this filter design problem is close to the optimal and quasi-optimal pool designs, which explains the similarities between them.

1 Maximizing the trace of the inverse input sample autocorrelation matrix: a filter design problem.

In this section, we formulate a filter design problem that intuitively maximizes the trace of the inverse sample autocorrelation matrix of the inputs. This matrix can be written as $\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}$, and therefore we want to study how to maximize the function

$$f(\mathbf{d}) \doteq \mathrm{Tr}\left\{\mathrm{E}\left\{(\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1}\right\}\right\} . \tag{5}$$

We first study the solutions of this problem when the input processes are i.i.d and follow a normal distribution $X_i^r \sim \mathrm{N}(0,1)$, $\forall r, i$. We then argue why the findings for this particular scenario may also apply in more general cases.

1.1 Normal input processes

Assume the input process are distributed as $\mathrm{N}(0,1)$, and both the number of users in the system N and the number of communication rounds observed ρ are sufficiently large. Also, note that matrix $\mathbf{D}$ is approximately circulant since we can disregard the border effects as $\rho \to \infty$. In that case, this matrix can be diagonalized by means of the DFT, i.e.,

$$\mathbf{D} \approx \mathbf{W}\mathbf{\Lambda}_{dd}\mathbf{W}^* , \tag{6}$$

where $\mathbf{W}$ is the $\rho \times \rho$ DFT matrix and $\mathbf{\Lambda}_{dd}$ is the diagonal matrix containing the ρ-point DFT of d_k. This means that $\mathbf{D}^T\mathbf{D} = \mathbf{W}|\mathbf{\Lambda}_{dd}|^2\mathbf{W}^*$. Therefore, we can write

$$f(\mathbf{d}) = \mathrm{Tr}\left\{\mathrm{E}\left\{(\mathbf{X}^T\mathbf{W}|\mathbf{\Lambda}_{dd}|^2\mathbf{W}^*\mathbf{X})^{-1}\right\}\right\} = \mathrm{Tr}\left\{\mathrm{E}\left\{(\tilde{\mathbf{X}}^H|\mathbf{\Lambda}_{dd}|^2\tilde{\mathbf{X}})^{-1}\right\}\right\} \tag{7}$$

where $\tilde{\mathbf{X}} \doteq \mathbf{W}^*\mathbf{X}$. Since the Wishart distribution is unitarily invariant and $\mathbf{W}$ is a unitary matrix, $\tilde{\mathbf{X}}$ has the same distribution as $\mathbf{X}$.

Given the expression in (7) and following equation (9) in [1], we can write the following identity for the trace function $f \equiv f(\mathbf{d})$:

$$\sum_{k=0}^{\rho-1} \frac{1}{1+\lambda_k \cdot f} = \rho - N \tag{8}$$

where λ_k is the k-th diagonal element of $|\mathbf{\Lambda}_{dd}|^2$. Note that this means that λ_k is the squared absolute value of the k-th frequency bin of the ρ-point DFT of the delay characteristic d_k, $k = 0, 1, \cdots, \rho - 1$.

Taking the derivative with respect to λ_i in the equation above, we get,

$$\frac{\partial f}{\partial \lambda_i} = -\frac{f}{(1+\lambda_i \cdot f)^2} \cdot \left(\sum_{k=0}^{\rho-1} \frac{\lambda_k}{(1+\lambda_k \cdot f)^2}\right) \tag{9}$$

Since $f > 0$ and $\lambda_k \geq 0$, this derivative is always negative, which implies that the trace increases when the values of λ_k decrease. In order to increase the trace as much as possible, one could think of making $\rho - N$ frequency bins close to zero, so that their contribution to the summation in (8) is close to $\rho - N$. This in turn would force the contribution of the rest of the terms in the sum to be close to zero, which means that $f \to \infty$. Therefore, this problem is equivalent to designing a filter d_k that minimizes the summation of a certain number of frequency bins.

We now explain intuitively the meaning of each constraint in this filter design problem. First of all, the constraint $\sum_{k=0}^{\rho-1} d_k = 1$ forces $\lambda_0 = 1$. The effects of the positivity constraints ($d_k \geq 0$, $\forall k$) in the filter design problem are complex and studied in the literature [2, 3, 4]. One of the consequences of these constraints is that $\lambda_0 \geq \lambda_k$ for $k \neq 0$. Another consequence, explained in detail in [3], is the fact that achieving an attenuation factor in high frequencies is much easier than increasing the magnitude of the filter around those frequencies, and therefore designing a low-pass filter is the best option if our aim is just to achieve low value of a certain number of frequency bins. One of the immediate consequences of the average delay constraint ($\sum_{k=1}^{\rho-1} k \cdot d_k \leq \bar{\delta}$) is that, from all the filters whose magnitude response (i.e., λ_k, $\forall k$) minimizes $f(\mathbf{d})$, we want the solution whose zeros lay inside the unit circle, i.e., the minimum phase filter. This is because the average delay of $\mathbf{d}$ and the group delay of the filter are closely related (note that they are equal if the delay characteristic is symmetric), and the minimum phase filter would be the one achieving a smaller average delay. We leave a deeper study of the effects of the delay constraint on this problem as subject for future work.

With all these considerations, we are ready to formulate an alternative optimization problem where we minimize the sum of the $\rho - N$ *high-frequency* values (assume N is even for simplicity):

$$\begin{aligned} \mathbf{d}''_{opt} = \underset{\mathbf{d}}{\operatorname{argmin}} \quad & \sum_{k=N/2+1}^{\rho-N/2+1} \lambda_k \\ \text{subject to} \quad & \sum_{k=0}^{\rho-1} d_k = 1, \quad d_k \geq 0, \ \forall k \\ & \sum_{k=1}^{\rho-1} k \cdot d_k \leq \bar{\delta} \end{aligned} \tag{10}$$

where λ_k is the k-th coefficient of the ρ-point DFT of d_k.

1.2 General input processes

The relation (8) above is only valid for normal inputs. We now explain why it is reasonable to apply the results of the previous section even if the inputs are not normal.

First, consider the diagonalization of $\mathbf{D}$ above in (6), together with the spectral decomposition

$$\mathbf{X}\mathbf{X}^T = \mathbf{A}\mathbf{\Lambda}_{xx}^2\mathbf{A}^T \tag{11}$$

where $\mathbf{A}$ is a $\rho \times \rho$ orthogonal matrix and $\mathbf{\Lambda}_{xx}^2$ is a diagonal matrix containing the eigenvalues of $\mathbf{X}\mathbf{X}^T$. Note that only N of these eigenvalues are non-zero. This leads to writing

$$\mathbf{X} = \mathbf{A}\mathbf{\Lambda}_{xx}\mathbf{S}\mathbf{B}^T \tag{12}$$

for some $N \times N$ orthogonal matrix $\mathbf{B}$ and a $\rho \times N$ column-selection matrix $\mathbf{S}$ that keeps the non-null columns of $\mathbf{\Lambda}_{xx}$.

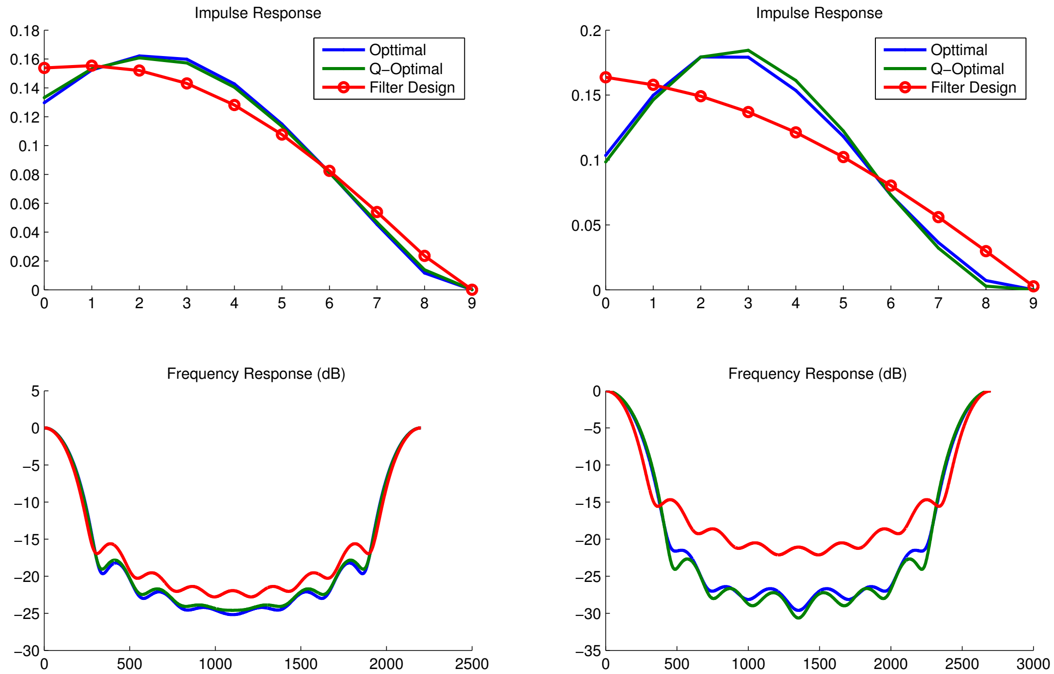
Let $\mathbf{V} \doteq \mathbf{W}^*\mathbf{A}$, and let $\mathbf{V}' = \mathbf{V}\mathbf{S}$. Also let $\mathbf{\Lambda}'_{xx}$ be the $N \times N$ diagonal matrix containing the N non-null eigenvalues of $\mathbf{X}\mathbf{X}^T$. Then, we can write the following chain of equalities:

$$\begin{aligned} (\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X})^{-1} &= (\mathbf{B}\mathbf{S}^T\mathbf{\Lambda}_{xx}\mathbf{A}^T\mathbf{W}|\mathbf{\Lambda}_{dd}|^2\mathbf{W}^*\mathbf{A}\mathbf{\Lambda}_{xx}\mathbf{S}\mathbf{B}^T)^{-1} && (13)\\ &= (\mathbf{B}\mathbf{\Lambda}'_{xx}\mathbf{S}^T\mathbf{A}^T\mathbf{W}|\mathbf{\Lambda}_{dd}|^2\mathbf{W}^*\mathbf{A}\mathbf{S}\mathbf{\Lambda}'_{xx}\mathbf{B}^T)^{-1} && (14)\\ &= (\mathbf{B}\mathbf{\Lambda}'_{xx}\mathbf{S}^T\mathbf{V}^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}\mathbf{S}\mathbf{\Lambda}'_{xx}\mathbf{B}^T)^{-1} && (15)\\ &= \mathbf{B}(\mathbf{\Lambda}'_{xx}\mathbf{S}^T\mathbf{V}^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}\mathbf{S}\mathbf{\Lambda}'_{xx})^{-1}\mathbf{B}^T && (16)\\ &= \mathbf{B}(\mathbf{\Lambda}'_{xx})^{-1}(\mathbf{S}^T\mathbf{V}^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}\mathbf{S})^{-1}(\mathbf{\Lambda}'_{xx})^{-1}\mathbf{B}^T && (17)\\ &= \mathbf{B}(\mathbf{\Lambda}'_{xx})^{-1}(\mathbf{V}'^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}')^{-1}(\mathbf{\Lambda}'_{xx})^{-1}\mathbf{B}^T && (18) \end{aligned}$$

In (13) we have just replaced $\mathbf{D}$ and $\mathbf{X}$ using (6) and (12). In (14) we have used that $\mathbf{\Lambda}_{xx}\mathbf{S} = \mathbf{S}\mathbf{\Lambda}'_{xx}$, and in (15) we have used $\mathbf{V} \doteq \mathbf{W}^*\mathbf{A}$. We have used the fact that $\mathbf{B}$ is orthogonal in (16) and that $\mathbf{\Lambda}'_{xx}$ is a non-singular diagonal matrix in (17). In the last step we have just used the definition of $\mathbf{V}'$.

Therefore, we can write the function (5) as

$$f(\mathbf{d}) = \operatorname{Tr}\left\{\operatorname{E}\left\{\mathbf{B}(\mathbf{\Lambda}'_{xx})^{-1}(\mathbf{V}'^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}')^{-1}(\mathbf{\Lambda}'_{xx})^{-1}\mathbf{B}^T\right\}\right\} . \tag{19}$$



(a) Threshold mix (b) Timed mix

Figure 1: Impulse and frequency responses of the optimal pool for the different problems, *Email* dataset.

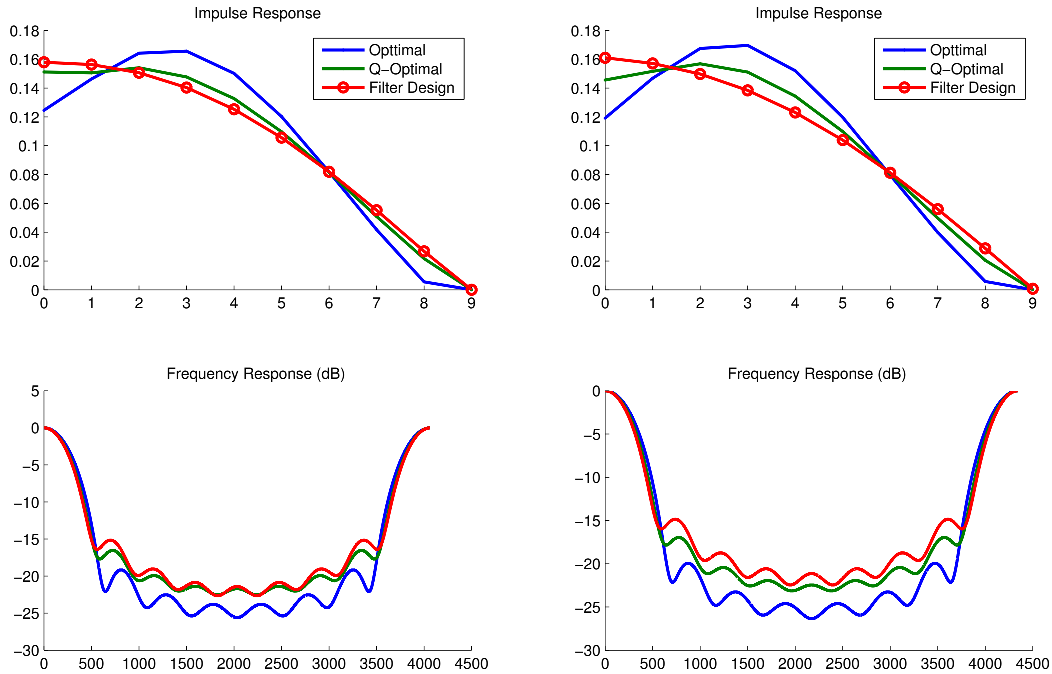
From random matrix theory, we know that "the eigenvectors of a random matrix are very likely to be delocalized in their sense that their l_2 energy is dispersed more or less evenly across its coefficients" [5]. This property would be satisfied by the elements of the columns of $\mathbf{A}$. The multiplication of $\mathbf{A}$ by the DFT matrix $\mathbf{W}^*$ to get $\mathbf{V} = \mathbf{W}^*\mathbf{A}$ is likely to increase the delocalization of the l_2 energy even further. We conjecture that because of this delocalization, the results obtained above for $(\tilde{\mathbf{X}}^H|\mathbf{\Lambda}_{dd}|^2\tilde{\mathbf{X}})^{-1}$ are also applicable to $(\mathbf{V}'^H|\mathbf{\Lambda}_{dd}|^2\mathbf{V}')^{-1}$, and therefore even if the input processes are not normal we can still model the optimal and quasi-optimal pool design problems as filter design problems. In the next section, we show that this is true for the real datasets we consider in our work.

2 Evaluation

We now show the solutions of the proposed problems (1), (4) and (10) for the real datasets introduced in the main document. We show the impulse response (i.e., the coefficients d_k) and the frequency response in dB (i.e., the values $10 \cdot \log_{10} \lambda_k$ defined above for $k = 0, \cdots, \rho - 1$). For the *Email* dataset, the results are shown in Figure 1. For *Location* and *MailingList*, the results are in Figs. 2 and 3 respectively.

As we can see, the solution of the filter design problem is not far from the optimal and quasi-optimal solutions in the threshold mix scenario, specially in the *MailingList* dataset. For the timed mix, the worst result is achieved in the *Email* dataset. If we look at the frequency response, we can see the optimal and quasi-optimal designs are indeed minimizing the sum of a reduced number of frequencies. Studying how to obtain the set of frequency bins that should be minimized to optimally reduce the trace $f(\mathbf{d})$ is left as subject for future work.

This shows that the problem of looking for the optimal and quasi-optimal pool can be seen as a filter design problem, where we want to find the delay characteristic $\mathbf{d}$ that reduces the sum of a set of high-frequency bins, given a set of constraints on $\mathbf{d}$. This explains the similarities between the optimal and quasi-optimal solutions.



(a) Threshold mix (b) Timed mix

Figure 2: Impulse and frequency responses of the optimal pool for the different problems, *Location* dataset.

References

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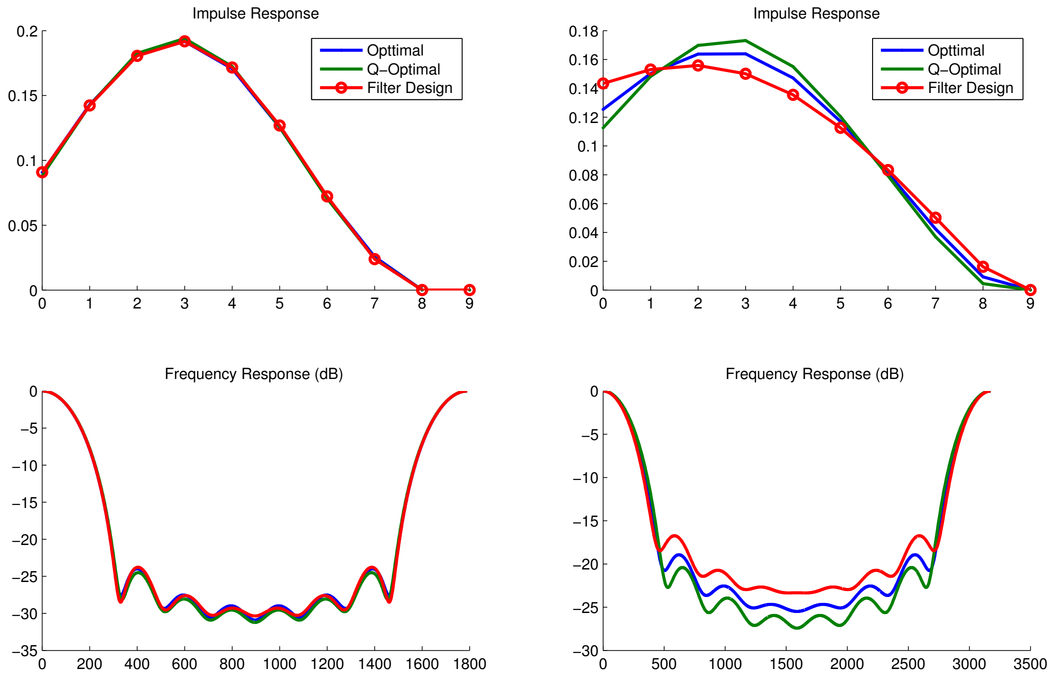

(a) Threshold mix (b) Timed mix

Figure 3: Impulse and frequency responses of the optimal pool for the different problems, *MailingList* dataset.