Second-order statistical properties of nonlinearly distorted phase-shift keyed (PSK) signals

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***Abstract*— In wireless communication systems operation of the amplifiers near saturation is often required for efficiency reasons, resulting in a nonlinearly distorted signal at the amplifier output. A popular model for the corresponding baseband equivalent nonlinear channel is a truncated Volterra series. By exploiting the bandpass nature of the channel and the statistical properties of phase-shift keyed signals, we show that the different terms in the Volterra series are white and uncorrelated with each other. This result is useful when considering blind equalization approaches for this class of systems.**

***Index Terms*— Nonlinear communication systems, Volterra series, Phase Shift Keying.**

I. Introduction

It is usual in wireless and satellite digital communication systems to operate the RF amplifier at or near saturation to improve its power efficiency [3]. The overall transmission system can then be modeled as in Fig. 1, by a memoryless analytic nonlinearity $f(\cdot)$ representing the near saturation amplifier characteristic, and a *bandpass* linear time invariant system with impulse response $c(t)$, comprising the effects of the bandpass transmit filter (which limits the spectrum of the output signal to the available bandwidth) and the propagation channel. The overall system thus induces *nonlinear* intersymbol interference (ISI).

Let the transmitted symbols be $a(k)$. In [1] it was shown that the sampled baseband equivalent nonlinear channel can be represented by a truncated Volterra series of the form

$$y(k) = \sum_{i=1}^{q} \sum_{j=0}^{l_i} h_{ij} s_i(k-j) + z(k). \quad (1)$$

Here $y(\cdot)$ is the sampled received signal, $z(\cdot)$ is the additive noise, and l_i is the degree of the i-th Volterra kernel. These kernels are defined as follows. The kernel indexed by $i=1$ is the linear kernel, i.e. $s_1(k) \triangleq a(k)$, the transmitted symbols. The terms $s_i(k)$ corresponding to the remaining kernels ($i > 1$) are restricted to be of the form

$$s_i(k) = \prod_{j=1}^{m_i} a^*(k - t_{ij}) \prod_{j=m_i+1}^{2m_i+1} a(k - t_{ij}), \quad (2)$$

This work was supported by NSF grants ECS-9970105, ECS-0225432 and CCR-9973133, and by a Ramón y Cajal grant of the Spanish Ministry of Science and Technology.

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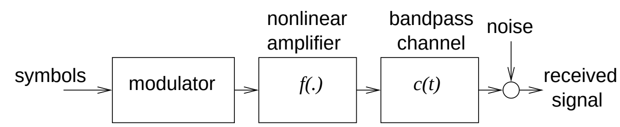

Fig. 1. Digital transmission system including a nonlinear amplifier.

where t_{ij} correspond to certain time delays and m_i are positive integers. Observe that in (2) only odd-order kernels appear, and that these have one more unconjugated input than conjugated inputs. Even-order distortions are absent due to the fact that they generate spectral components which lie outside the channel bandwidth (centered at the carrier frequency) and therefore are rejected by the bandpass filter following the amplifier [1]. The h_{ij} are the coefficients of the i-th baseband equivalent Volterra kernel, which could be $N \times 1$ vectors if some form of diversity is available resulting in N subchannels. The parameters h_{ij}, l_i, m_i and t_{ij} can be determined from the Taylor series expansion of the amplifier nonlinearity and the impulse response $c(t)$ [1]. Analyticity of $f(\cdot)$ guarantees the existence of its Taylor series expansion. As with all Volterra series models, the fidelity with which (1)-(2) represent the actual system improves with larger q, the total number of kernels.

Motivated by considerations of blind equalization of channels such as these, when both the impulse response $c(t)$ and the nonlinearity $f(\cdot)$, and hence its Taylor series coefficients, are potentially unknown at the receiver, in this letter we are concerned with the second-order statistical properties of the terms $s_i(\cdot)$ in (2). In particular, we have presented in [4] sufficient conditions under which, both zero forcing and minimum mean square error equalizers for channels represented by truncated Volterra series, can be obtained from the output second-order statistics alone. One such condition appearing in the analysis is that

$$\text{cov}[s_i(k), s_j(k-d)] \neq 0 \quad \text{iff} \quad i = j \quad \text{and} \quad d = 0. \quad (3)$$

i.e. each $s_i(\cdot)$ is a white process, and for $i \neq j$ the two processes $s_i(\cdot)$, $s_j(\cdot)$ are uncorrelated.

We note that in general, even if the transmitted symbols $a(\cdot)$ are zero mean, independent, identically distributed (iid), the nonlinearities implicit in (2) may prevent (3) from holding, even though the unknown nonlinearity $f(\cdot)$ is memoryless. The contribution of this letter is to show that should the process $a(\cdot)$ be iid and drawn equiprobably from a PSK constellation, then because only odd order kernels appear in the Volterra expansion, and symmetries manifest in PSK constellations [see e.g. (5) and (6)], (3) does indeed hold.

II. Main result

We make the following assumption:

Assumption 1: The symbols $a(\cdot)$ are drawn independently and with identical probabilities from an M-ary PSK constellation: $a(k) \in \{R \cdot e^{j2\pi n/M}, \quad n = 0, 1, \ldots, M-1\}$.

Our main result can then be stated as follows.

Theorem 1: Consider the terms $s_i(\cdot)$ in (2). Then under Assumption 1, (3) holds.

Proof: Given two terms $s_i(k)$, $s_j(k)$ of the form (2), they can be written as

$$\begin{aligned} s_i(k) &= a^{p_0}(k) a^{p_1}(k-1) \cdots a^{p_t}(k-t) \\ &\quad \times [a^{p'_0}(k)]^* [a^{p'_1}(k-1)]^* \cdots [a^{p'_t}(k-t)]^*, \\ s_j(k) &= a^{q_0}(k) a^{q_1}(k-1) \cdots a^{q_t}(k-t) \\ &\quad \times [a^{q'_0}(k)]^* [a^{q'_1}(k-1)]^* \cdots [a^{q'_t}(k-t)]^*, \end{aligned}$$

for some integers t and p_i, p'_i, q_i, q'_i such that

$$(p_0 - p'_0) + \cdots + (p_t - p'_t) = 1, \quad (q_0 - q'_0) + \cdots + (q_t - q'_t) = 1. \tag{4}$$

(Some p_i, p'_i, q_i, q'_i may be zero). We shall show that if $\mathrm{cov}[s_i(k), s_j(k-d)] \neq 0$ for some d, then $s_i(k)$ must be a scaled version of $s_j(k-d)$. To do so, note that for all integers $l > 0$, the circular symmetry of the M-ary PSK constellation gives

$$\begin{aligned} E[a^l(k)] &= \frac{R^l}{M} \sum_{n=0}^{M-1} \mathrm{e}^{j2\pi nl/M} = \frac{R^l}{M} \frac{1 - \mathrm{e}^{j2\pi l}}{1 - \mathrm{e}^{j2\pi l/M}} \\ &= \begin{cases} 0, & l \bmod M \neq 0, \\ R^l, & l \bmod M = 0. \end{cases} \end{aligned} \tag{5}$$

But if $l \bmod M = 0$ then $a^l(k) = R^l$ reduces to a constant. Thus without loss of generality we can assume that none of the p_i, p'_i, q_i, q'_i are multiples of M, since any multiplicative constants in the terms $s_i(k)$, $s_j(k)$ can be absorbed by the channel coefficients. As a consequence, since

$$E\left\{a^l(k)[a^{l'}(k)]^*\right\} = \begin{cases} R^{2l'} E\left\{a^{l-l'}(k)\right\}, & l > l', \\ R^{2l}, & l = l', \\ R^{2l} E\left\{[a^{l'-l}(k)]^*\right\}, & l < l', \end{cases} \tag{6}$$

we have that for all l, l' with $l, l' \bmod M \neq 0$,

$$E\left\{a^l(k)[a^{l'}(k)]^*\right\} \neq 0 \qquad \text{only if } l = l', \tag{7}$$

in which case the term $a^l(k)[a^l(k)]^*$ is constant:

$$a^l(k)[a^l(k)]^* = R^{2l}. \tag{8}$$

Now observe that due to the independence and stationarity of the symbols $a(\cdot)$, one has

$$E[s_i(k)] = \prod_{n=0}^{t} E\left\{a^{p_n}(k)[a^{p'_n}(k)]^*\right\},$$

which is zero in view of (7) since (4) precludes having $p_j = p'_j$ for all $0 \leq j \leq t$; similarly, $E[s_j(k)] = 0$. Hence the terms have zero mean, so that

$$\mathrm{cov}[s_i(k), s_j(k-d)] = E[s_i(k) s_j^*(k-d)].$$

Suppose that $E[s_i(k) s_j^*(k-d)] \neq 0$. Then we must have $0 \leq d \leq t$, or otherwise $s_i(k)$, $s_j(k-d)$ are independent and their covariance becomes zero automatically. Note that

$$\begin{aligned} E[s_i(k) s_j^*(k-d)] &= E\left\{\prod_{n=0}^{d-1} a^{p_n}(k-n)[a^{p'_n}(k-n)]^*\right\} \\ &\quad \times E\left\{\prod_{n=d}^{t} a^{p_n}(k-n)[a^{p'_n}(k-n)]^* \right. \\ &\qquad \left. a^{q'_{n-d}}(k-n)[a^{q_{n-d}}(k-n)]^*\right\} \\ &\quad \times E\left\{\prod_{n=t+1}^{t+d} a^{q'_{n-d}}(k-n)[a^{q_{n-d}}(k-n)]^*\right\} \end{aligned} \tag{9}$$

If (9) is nonzero, then the three factors in the right hand side must be nonzero. Using the stationarity and iid properties of the symbol sequence, for the first factor one has

$$\begin{aligned} 0 &\neq \prod_{n=0}^{d-1} E\left\{a^{p_n}(k)[a^{p'_n}(k)]^*\right\} \\ &\Rightarrow \quad p_n = p'_n, \qquad 0 \leq n \leq d-1, \end{aligned} \tag{10}$$

in view of (7). Similarly, for the third factor,

$$\begin{aligned} 0 &\neq \prod_{n=t-d+1}^{t} E\left\{a^{q'_n}(k)[a^{q'_n}(k)]^*\right\} \\ &\Rightarrow \quad q_n = q'_n, \qquad t-d+1 \leq n \leq t. \end{aligned} \tag{11}$$

From (8), (10) and (11), it follows that we must have

$$\begin{aligned} s_i(k) &= R^{2(p_0 + \cdots + p_{d-1})} \prod_{n=d}^{t} a^{p_n}(k-n)[a^{p'_n}(k-n)]^*, \\ s_j(k) &= R^{2(q_{t-d+1} + \cdots + q_t)} \prod_{n=0}^{t-d} a^{q_n}(k-n)[a^{q'_n}(k-n)]^*. \end{aligned}$$

In addition, the second factor in the right hand side of (9) must be nonzero. Therefore

$$\begin{aligned} 0 &\neq \prod_{n=d}^{t} E\left\{a^{p_n}(k) a^{q'_{n-d}}(k)[a^{p'_n}(k)]^*[a^{q_{n-d}}(k)]^*\right\} \\ &\Rightarrow \quad p_n - p'_n = q_{n-d} - q'_{n-d}, \quad d \leq n \leq t. \end{aligned} \tag{12}$$

Define now $\bar{p}_n \triangleq \min\{p_n, p'_n\}$, $\bar{q}_n \triangleq \min\{q_n, q'_n\}$, and let

$$b_n(k) \triangleq \begin{cases} a(k), & p_n \geq p'_n, \\ a^*(k), & p_n < p'_n, \end{cases} \tag{13}$$

$$c_n(k) \triangleq \begin{cases} a(k), & q_n \geq q'_n, \\ a^*(k), & q_n < q'_n. \end{cases} \tag{14}$$

Then one has

$$\begin{aligned} a^{p_n}(k)[a^{p'_n}(k)]^* &= R^{2\bar{p}_n}[b_n(k)]^{|p_n - p'_n|}, \\ a^{q_n}(k)[a^{q'_n}(k)]^* &= R^{2\bar{q}_n}[c_n(k)]^{|q_n - q'_n|}. \end{aligned}$$

Then we can write

$$\begin{aligned} s_i(k) &= R^{2\left(\sum_{n=0}^{d-1} p_n + \sum_{n=d}^{t} \bar{p}_n\right)} \prod_{n=d}^{t} [b_n(k-n)]^{|p_n - p'_n|}, \\ s_j(k) &= R^{2\left(\sum_{n=0}^{t-d} \bar{q}_n + \sum_{n=t-d+1}^{t} q_n\right)} \prod_{n=0}^{t-d} [c_n(k-n)]^{|q_n - q'_n|}. \end{aligned}$$

But in view of (12)-(14) it follows that for $d \leq n \leq t$, $p_n - p'_n = q_{n-d} - q'_{n-d}$ and $b_n(k) = c_{n-d}(k)$. Therefore $s_i(k) = \bar{c} \cdot s_j(k-d)$ where $\bar{c}$ is a constant, as was to be shown. ■

III. Extension to other modulation schemes

The proof of Theorem 1 critically exploits the symmetry of the PSK constellation and therefore it does not immediately apply to other constellations such as higher-order quadrature amplitude modulation (QAM) for which this symmetry is lost. However, for certain PSK-derived modulation schemes commonly found in wireless applications, Theorem 1 is still valid:

1) *Offset Quadrature PSK (OQPSK):* This method is similar to QPSK, but the in-phase and quadrature components of the transmitted signal are offset in their relative time alignment by half a symbol period, effectively constraining the maximum phase shift to $\pm 90°$. The symbols $a(k)$ can be seen as being drawn from $\{-1, +1\}$ for even k and from $\{-j, +j\}$ for odd k [1].
2) $\pi/4$ *QPSK:* In this scheme the symbols are alternatively drawn from two different PSK constellations with $M = 4$ (QPSK) with one of them rotated $\pi/4$ radians with respect to the other. This constrains the maximum phase shift to $\pm 135°$ [2].

These signaling methods are preferred to QPSK in systems with nonlinear amplifiers [1], [2]. In both cases, the symbols belong to constellations that vary from symbol to symbol but which have the desired symmetry properties.

IV. Conclusions

The second-order statistical properties of the terms appearing in the Volterra series expansion of a bandpass nonlinearity have been analyzed. When the signaling method is phase-shift keying (PSK) or one of its variants, we have shown that these terms are white and uncorrelated. Application areas include blind equalization of nonlinear channels.

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