Frequency Estimation of Real-Valued Single-Tone in Colored Noise Using Multiple Autocorrelation Lags

[1]Rim Elasmi-Ksibi, [1]Hichem Besbes, [2]Roberto López-Valcarce and [1]Sofiane Cherif

Abstract

The problem of frequency estimation of a real-valued sinusoid in colored noise is addressed. A new estimator using higher-order lags of the sample autocorrelation is proposed, and a statistical analysis is provided. By choosing the number of lags to include in the estimation process, a tradeoff between performance and complexity can be easily achieved.

I. Introduction

Detection of sinusoidal components and estimation of their frequencies in the presence of broadband noise are common problems in signal processing with a broad range of areas of application; numerous techniques have been developed for the white noise case [1]. The Maximum Likelihood (ML) method is statistically efficient, achieving the Cramer-Rao Lower Bound (CRLB) asymptotically, but it is computationally demanding [2]. Simpler, suboptimal frequency estimators can be obtained using the Linear Prediction properties of sinusoidal signals, such as the Pisarenko Harmonic Decomposer (PHD) [3], Reformed PHD [4] and Modified Covariance (MC) [5] methods; a number of statistical analyses have shown their inefficiency [6], [7].

For the single sinusoid case, these simple estimators make use of just two coefficients of the sample autocorrelation. Using multiple autocorrelation lags has been considered in [8], [9] for the complex-valued (cisoid) case, as well as in [10] (the so-called P-estimator) for the real-valued case. Although these approaches result in performance improvement, they require phase unwrapping to resolve the

(1) Unité de Recherche TECHTRA, Sup'Com, Tunisia

(2) Dept. Signal Theory and Communications, University of Vigo, Spain

e-mail: asmi.rim@planet.tn, {hichem.besbes, sofiane.cherif}@supcom.rnu.tn, valcarce@gts.tsc.uvigo.es

This work was supported by the Spanish Agency for International Cooperation (AECI) and the Tunisian Government under project A/5405/06, and by the Spanish Ministry of Science and Innovation under projects SPROACTIVE (ref. TEC2007-68094-C02-01/TCM) and COMONSENS (CONSOLIDER-INGENIO 2010 CSD2008-00010).

frequency ambiguities that then appear. The phase unwrapping stage adds extra complexity to the procedure, which is clearly undesirable in applications requiring rapid frequency estimation.

We focus on the real-valued case, and present a novel estimator that exploits higher lags of the sample autocorrelation. It enjoys low computational complexity since no phase unwrapping is required. The new method compares favorably to its forerunners, and it is applicable to environments with colored noise of finite memory (e.g. Moving Average (MA) noise models).

Section II states the problem and presents the new estimator. A statistical analysis is given in Section III. Section IV shows some numerical examples, and conclusions are drawn in Section V.

II. Novel frequency estimator

Consider the problem of estimating the unknown frequency $\omega_0 \in [0, \pi]$ of a real-valued sine wave s_n immersed in colored noise u_n. The N available samples of the observed signal, y_n, are given by

$$\begin{aligned} y_n &= s_n + u_n \\ &= \alpha \sin(\omega_0 n + \varphi) + u_n, \qquad 1 \leq n \leq N, \end{aligned} \tag{1}$$

where α is the sinusoid amplitude and φ is a random phase uniformly distributed in $[-\pi, \pi[$. The noise process $\{u_n\}$ is zero-mean wide-sense stationary and Gaussian, and independent of $\{s_n\}$. It is modeled as an MA(M) process, generated by passing a white Gaussian process with variance σ^2 through an order-M FIR filter. The noise autocorrelation is denoted $\mathrm{E}\{u_n u_{n-k}\} = \sigma^2 \rho_k$; thus, $\rho_k = 0$ for $k > M$. The power spectral density of $\{u_n\}$ is $S_u(e^{j\omega}) = \sigma^2 \sum_{k=-M}^{M} \rho_k e^{-jk\omega}$. It will also be useful to define $\beta_l \doteq \sum_{k=-M}^{M} \rho_k \rho_{k-l}$. The Signal to Noise Ratio is defined as SNR $\doteq \alpha^2/(2\sigma^2)$.

The autocorrelation sequence of the observations is

$$r_k \doteq \mathrm{E}\{y_n y_{n-k}\} = \frac{\alpha}{2} \cos k\omega_0 + \sigma^2 \rho_k. \tag{2}$$

Thus, using trigonometric relations one can show that

$$r_{k-1} + r_{k+1} = 2 r_k \cos \omega_0, \qquad \text{for all } k > M+1. \tag{3}$$

Hence, we observe that for any integers $q > p > M+1$,

$$\sum_{k=p}^{q} r_k (r_{k-1} + r_{k+1}) = 2 \cos \omega_0 \sum_{k=p}^{q} r_k^2. \tag{4}$$

Eq. (4) suggests a means to obtain an estimate $\hat{a}_0 \doteq \cos \hat{\omega}_0$ as

$$\hat{a}_0 = \frac{\sum_{k=p}^{q} \hat{r}_k (\hat{r}_{k-1} + \hat{r}_{k+1})}{2 \sum_{k=p}^{q} \hat{r}_k^2}, \tag{5}$$

where $\hat{r}_k$ is the unbiased autocorrelation estimate

$$\hat{r}_k \doteq \frac{1}{N-k} \sum_{n=k+1}^{N} y_n y_{n-k}. \tag{6}$$

III. PERFORMANCE ANALYSIS

Let $\hat{\boldsymbol{r}} \doteq [\ \hat{r}_{p-1} \quad \hat{r}_p \quad \cdots \quad \hat{r}_{q+1}\ ]^T$. We can express the novel estimator as $\hat{a}_0 = f(\hat{\boldsymbol{r}})$, where f is defined by (5). A first-order Taylor expansion of $f(\hat{\boldsymbol{r}})$ around $\boldsymbol{r} \doteq [\ r_{p-1} \quad r_p \quad \cdots \quad r_{q+1}\ ]^T$ yields

$$\hat{a}_0 \approx a_0 + \boldsymbol{v}^T(\hat{\boldsymbol{r}} - \boldsymbol{r}), \tag{7}$$

with $\boldsymbol{v} \doteq \nabla f(\boldsymbol{r})|_{\boldsymbol{r}=\hat{\boldsymbol{r}}}$. Thus, $\mathrm{E}\{(\hat{a}_0 - a_0)^2\} \approx \boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v}$, with $\boldsymbol{C}$ the covariance matrix of $\hat{\boldsymbol{r}}$. One has

$$\boldsymbol{v} = \frac{1}{\alpha^2\Delta(\omega_0)}\left[\ \cos p\omega_0 \quad -\cos(p-1)\omega_0 \quad 0 \quad \cdots \quad 0 \quad -\cos(q+1)\omega_0 \quad \cos q\omega_0\ \right]^T, \tag{8}$$

with $\Delta(\omega) \doteq \sum_{k=p}^{q}\cos^2 k\omega$. It is shown in the Appendix that

$$\boldsymbol{C} \approx \frac{1}{N}\left[\sigma^4\boldsymbol{B} + 2\alpha^2 S_u(e^{j\omega_0})\boldsymbol{w}\boldsymbol{w}^T\right], \tag{9}$$

with $\boldsymbol{B}$ a Toeplitz matrix with elements $[\boldsymbol{B}]_{i,j} = \beta_{i-j}$, and $\boldsymbol{w}$ a vector with elements $[\boldsymbol{w}]_i = \cos(p+i-2)\omega_0$. It turns out that $\boldsymbol{v}^T\boldsymbol{w} = 0$, and thus, using (8)-(9) and $\mathrm{E}\{(\hat{\omega}_0-\omega_0)^2\} \approx \mathrm{E}\{(\hat{a}_0-a_0)^2\}/\sin^2\omega_0$, one obtains

$$\mathrm{E}\{(\hat{\omega}_0 - \omega_0)^2\} \approx \frac{\Gamma(\omega_0)}{4N\mathrm{SNR}^2\Delta^2(\omega_0)\sin^2\omega_0}, \tag{10}$$

where $\Gamma(\omega) = \sum_{i\in\mathcal{I}}\beta_i\Gamma_i(\omega)$, with $\mathcal{I} = \{0, 1, q-p, q-p+1, q-p+2\}$ and

$$\Gamma_0(\omega) \doteq \cos^2(p-1)\omega + \cos^2 p\omega + \cos^2 q\omega + \cos^2(q+1)\omega \tag{11}$$

$$\Gamma_1(\omega) \doteq -2[\cos(p-1)\omega\cos p\omega + \cos q\omega\cos(q+1)\omega] \tag{12}$$

$$\Gamma_{q-p}(\omega) \doteq 2\cos(p-1)\omega\cos(q+1)\omega \tag{13}$$

$$\Gamma_{q-p+1}(\omega) \doteq -2[\cos p\omega\cos(q+1)\omega + \cos(p-1)\omega\cos q\omega] \tag{14}$$

$$\Gamma_{q-p+2}(\omega) \doteq 2\cos p\omega\cos q\omega. \tag{15}$$

Note that if q, p are chosen such that $q-p > 2M$, then $\beta_{q-p} = \beta_{q-p+1} = \beta_{q-p+2} = 0$. For the case of white Gaussian noise, $\beta_l = \delta_l$ and thus the Mean Square Error (MSE) of the estimator is

$$\mathrm{E}\{(\hat{\omega}_0 - \omega_0)^2\} \approx \frac{\cos^2(p-1)\omega_0 + \cos^2 p\omega_0 + \cos^2 q\omega_0 + \cos^2(q+1)\omega_0}{4N\mathrm{SNR}^2\left(\sum_{k=p}^{q}\cos^2 k\omega_0\right)^2\sin^2\omega_0}. \tag{16}$$

The influence of the design parameters p, q in the MSE is via the factor $\Gamma(\omega_0)/\Delta^2(\omega_0)$. Although this term is highly oscillatory, the amplitude of the oscillations decreases as q increases. It is instructive to consider (16) for $\omega_0 = \frac{\pi}{2}$, for which $\Gamma(\omega_0) = 2$ and $\Delta(\omega_0) \approx (q-p+1)/2$; thus, the MSE behaves as $1/(q-p+1)^2$. Roughly speaking, by doubling the number of lags included in the estimation process, the MSE is decreased by 6 dB. Of course, this reduction of the MSE cannot be achieved indefinitely by increasing q, since the finite sample size limits the number of autocorrelation estimates available; at the same time, the approximations used when deriving (10) become less accurate with larger q.

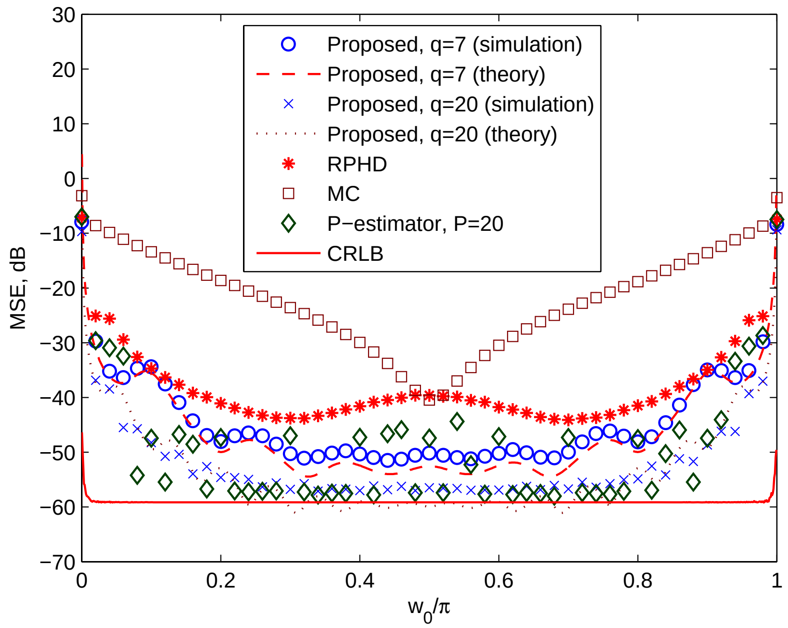

Fig. 1. MSE vs. ω_0. White noise case, SNR = 10 dB, $N = 100$, $p = 2$.

IV. Simulation results

Fig. 1 shows the MSE of several estimators in a white noise setting as a function of ω_0, for $N = 100$ and SNR = 10 dB. The RPHD [4] and MC [5] schemes use two autocorrelation coefficients only, performing worse than methods exploiting higher-order lags. Among these, the P-estimator [10] suffers from the so-called 'edge frequency' problem: performance degrades for ω_0 close to $k\pi/P$, with $0 \le k \le P-1$ ($P = 20$ in this example). For the new estimator (5), $p = 2$ was taken, and two different values of q (7 and 20) were considered. As expected, performance improves with increasing q, since more autocorrelation lags are then included in the estimation process. Good agreement with the approximate MSE (16) is observed. With $q = 20$, the novel estimator is very close to the CRLB [11] for a wide range of frequencies.

Next we consider a colored noise case in which the noise coloring filter has transfer function $B(z) = 1 + 0.1z^{-1} + 0.7z^{-2} + 0.05z^{-3} + 0.3z^{-4}$. Since $M = 4$, now $p = 6 > M + 1$ is adopted for the novel estimator. The RPHD and MC estimates are biased in the presence of noise coloring, resulting in the poor MSE behavior observed in Fig. 2. The novel estimator still offers a means to trade off performance and complexity by increasing q. Fig. 3 shows the MSE of the estimators as a function of the SNR, fixing $\omega_0 = 0.4\pi$. Using $(p, q) = (6, 30)$, the novel estimate asymptotically achieves the CRLB.

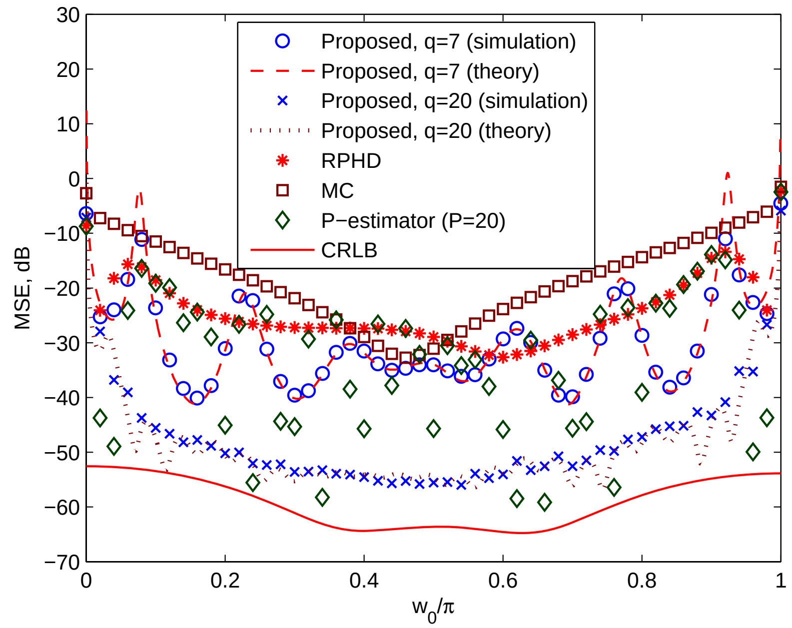

Fig. 2. MSE vs. ω_0. Colored noise case, SNR = 10 dB, $N = 100$, $p = 6$.

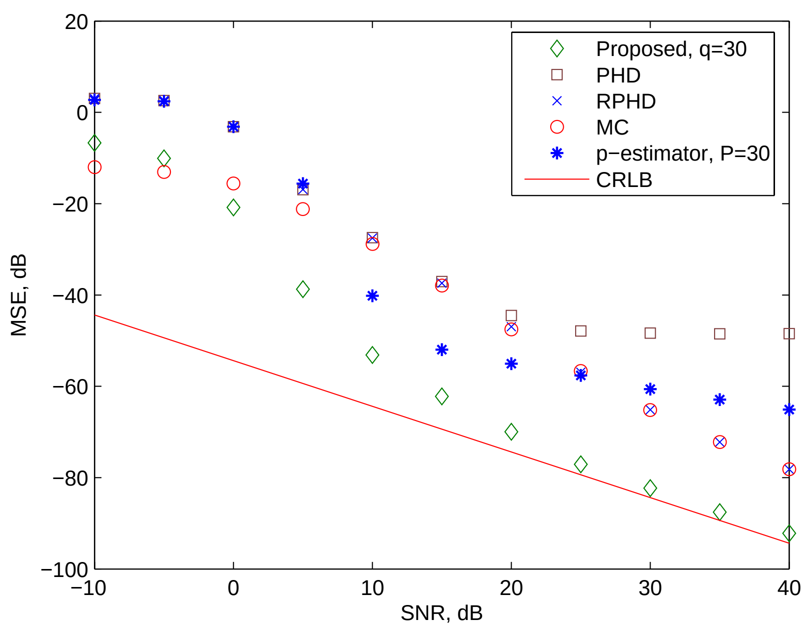

Fig. 3. MSE vs. SNR. Colored noise case, $\omega = 0.4\pi$, $N = 100$, $q = 30$, $p = 6$.

V. Conclusion

A frequency estimator for real-valued sinusoids in noise exploiting higher-order lags of the sample autocorrelation has been presented. For MA noise, if an upper bound is available for the order of the noise coloring filter, the ’corrupted’ low-order lags can be left out, and thus bias is avoided. Knowledge of the MA noise model parameters is not needed. The proposed method compares favorably to previous approaches and allows a tradeoff between performance and complexity by choosing the number of lags to be included in the estimation process.

APPENDIX

Let k, $m > M$. One has $\mathrm{E}\{(\hat{r}_k - r_k)(\hat{r}_m - r_m)\} = \mathrm{E}\{\hat{r}_k\hat{r}_m\} - r_m r_k$, and

$$\begin{aligned} \mathrm{E}\{\hat{r}_k\hat{r}_m\} &= \frac{1}{N^2}\sum_{i=k+1}^{N}\sum_{j=m+1}^{N}\mathrm{E}\{y_i y_j y_{i-k} y_{j-m}\} \\ &= \frac{\alpha^4}{4}\cos k\omega_0 \cos m\omega_0 + T_0 + T_1, \end{aligned} \tag{17}$$

where T_0 and T_1 are defined as

$$\begin{aligned} T_0 \doteq \frac{1}{N^2}\sum_{i=k+1}^{N}\sum_{j=m+1}^{N}\Big[&\mathrm{E}\{s_{i-k}s_{j-m}\}\mathrm{E}\{u_i u_j\} + \mathrm{E}\{s_i s_j\}\mathrm{E}\{u_{i-k}u_{j-m}\} \\ &+ \mathrm{E}\{s_i s_{j-m}\}\mathrm{E}\{u_{i-k}u_j\} + \mathrm{E}\{s_{i-k}s_j\}\mathrm{E}\{u_i u_{j-m}\} \\ &+ \mathrm{E}\{s_i s_{i-k}\}\underbrace{\mathrm{E}\{u_j u_{j-m}\}}_{=0 \text{ for } m>M} + \mathrm{E}\{s_j s_{j-m}\}\underbrace{\mathrm{E}\{u_i u_{i-k}\}}_{=0 \text{ for } k>M}\Big], \end{aligned} \tag{18}$$

$$T_1 \doteq \frac{1}{N^2}\sum_{i=k+1}^{N}\sum_{j=m+1}^{N}\mathrm{E}\{u_i u_j u_{i-k} u_{j-m}\}. \tag{19}$$

Assume w.l.o.g. that $k \geq m$, and define a trapezoidal window w_l as

$$w_l = \begin{cases} N-k-|l|, & -N+k+1 \leq l \leq 0, \\ N-k, & 0 < l \leq k-m, \\ N-m-|l|, & k-m < l \leq N-m-1. \end{cases} \tag{20}$$

Then, for large N, the following approximation holds:

$$\begin{aligned} \frac{1}{N^2}\sum_{i=k+1}^{N}\sum_{j=m+1}^{N}\mathrm{E}\{s_{i-k}s_{j-m}\}\mathrm{E}\{u_i u_j\} &= \frac{1}{N^2}\sum_{l=k+1-N}^{N-m-1} w_l \mathrm{E}\{s_{n-k}s_{n+l-m}\}\mathrm{E}\{u_n u_{n+l}\} \\ &\approx \frac{1}{N}\sum_{l=-\infty}^{\infty}\mathrm{E}\{s_{n-k}s_{n+l-m}\}\mathrm{E}\{u_n u_{n+l}\} \\ &= \frac{1}{N}\sum_{l=-\infty}^{\infty}\frac{\alpha^2}{2}\cos(l-m+k)\omega_0 \cdot \sigma^2\rho_l \\ &= \frac{1}{N}\frac{\alpha^2}{2}\cos(k-m)\omega_0 S_u(e^{j\omega_0}). \end{aligned}$$

The remaining terms in (18) are obtained analogously, yielding

$$T_0 \approx \frac{2\alpha^2}{N}S_u(e^{j\omega_0})\cos k\omega_0 \cos m\omega_0. \tag{21}$$

Now, since $\{u_n\}$ is Gaussian,

$$\mathrm{E}\{u_i u_j u_{i-k} u_{j-m}\} = \mathrm{E}\{u_i u_j\}\mathrm{E}\{u_{i-k}u_{j-m}\} + \mathrm{E}\{u_i u_{j-m}\}\mathrm{E}\{u_{i-k}u_j\} + \underbrace{\mathrm{E}\{u_i u_{i-k}\}}_{=0 \text{ for } k>M}\underbrace{\mathrm{E}\{u_j u_{j-m}\}}_{=0 \text{ for } m>M}, \tag{22}$$

and therefore

$$
\begin{aligned}
T_1 &= \frac{1}{N^2}\sum_{i=k+1}^{N}\sum_{j=m+1}^{N} \mathrm{E}\{u_i u_j\}\mathrm{E}\{u_{i-k}u_{j-m}\} + \mathrm{E}\{u_i u_{j-m}\}\mathrm{E}\{u_{i-k}u_j\} \\
&= \frac{\sigma^4}{N^2}\sum_{l=k+1-N}^{N-m-1} w_l(\rho_l\rho_{l-k+m} + \rho_{l+m}\rho_{l-k}) \\
&\approx \frac{\sigma^4}{N}\sum_{l=-M}^{M}\rho_l(\rho_{l-k+m} + \rho_{l-k-m}) = \frac{\sigma^4}{N}\sum_{l=-M}^{M}\rho_l\rho_{l-k+m},
\end{aligned}
\tag{23}
$$

where the last step follows from the fact that $l-k-m<-M$.

REFERENCES

[1] P. Stoica and R. L. Moses, *Spectral Analysis of Signals*. Prentice Hall, 2005.

[2] R. J. Kenefic and A. H. Nuttall, “Maximum likelihood estimation of the parameters of tone using real discrete data,” *IEEE J. Oceanic Eng.*, vol. OE-12, no. 1, pp. 279–280, 1987.

[3] V. F. Pisarenko, “The retrieval of harmonics by linear prediction,” *Geophys. J. R. Astron. Soc*, vol. 33, pp. 347–366, 1973.

[4] H. C. So and K. W. Chan, “Reformulation of Pisarenko harmonic decomposition method for single-tone frequency estimation,” *IEEE Trans. on Signal Processing*, vol. 52, pp. 1128–1135, Apr. 2004.

[5] L. B. Fertig and J. H. McClellan, “Instantaneous frequency estimation using linear prediction with comparisons to the DESAs,” *IEEE Signal Processing Lett.*, vol. 3, pp. 54–56, Feb. 1996.

[6] H. Sakai, “Statistical analysis of Pisarenko’s method for sinusoidal frequency estimation,” *IEEE Trans. Acous., Speech, Signal Processing*, vol. ASSP-32, pp. 95–101, Feb 1984.

[7] A. Eriksson and P. Stoica, “On statistical analysis of Pisarenko tone frequency estimator,” *Signal Processing*, vol. 31, pp. 349–353, Apr. 1993.

[8] B. Völcker and P. Händel, “Frequency estimation from proper sets of correlations,” *IEEE Transactions on Signal Processing*, vol. 50, no. 4, pp. 791–802, 2002.

[9] Z. Wang and S. S. Abeysekera, “Frequency estimation from multiple lags of correlations in the presence of MA colored noise,” *Proc. ICASSP*, pp. 693–696, 2005.

[10] R. Elasmi-Ksibi, R. López-Valcarce, H. Besbes, and S. Cherif, “A family of real single-tone frequency estimators using higher-order sample covariance lags,” *Proc. European Signal Processing Conf. (EUSIPCO)*, pp. 956–959, 2008.

[11] D. N. Swingler, “Approximate bounds on frequency estimates for short cisoids in colored noise,” *IEEE Trans. on Signal Processing*, vol. 46, p. 14561458, May 1998.