

# GENERALIZED MATCHED FILTER DETECTOR FOR FAST FADING CHANNELS

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## ABSTRACT

We consider the problem of detecting a known signal with constant magnitude immersed in noise of unknown variance, when the propagation channel is frequency-flat and randomly time-varying within the observation window. A Basis Expansion Model with random coefficients is used for the channel, and a Generalized Likelihood Ratio approach is adopted in order to cope with deterministic nuisance parameters. The resulting scheme can be seen as a generalization of the well-known Matched Filter detector, to which it reduces for time-invariant channels. Closed-form analytical expressions are provided for the distribution of the test statistic under both hypotheses, which allow to assess the detection performance.

**Index Terms**— Flat fading channels, detection, Generalized Likelihood Ratio, Matched Filter.

## 1. INTRODUCTION

Signal activity detection in unknown channels plays an important role in many signal processing applications, such as sonar [1], radar [2] and spectrum sensing [3] among others. Most detectors from the literature assume that the channel changes sufficiently slowly to allow for a block-fading model. However, applications exist in which this assumption becomes unrealistic, so that channel variations within the observation window should be taken into account. In narrowband wireless communication systems, for example, the coherence time of the channel may be comparable to the symbol period. In spectrum sensing applications, operation in very low Signal-to-Noise Ratio (SNR) conditions [3] requires long observation intervals which may invalidate the time-invariant assumption. The large Doppler spreads found in underwater communication systems will also give rise to fast channel variations [1].

We address the problem of deciding on the presence of a known constant magnitude waveform in noise of unknown variance, when the propagation channel is frequency-flat and randomly time-varying. To this end, the low-pass nature of

the Doppler spectrum of typical channels found in practice will be exploited by resorting to a Basis Expansion Model (BEM) for the channel time variations [4, 5]. Whereas the majority of previous work on fast fading channels has focused on modeling, estimation, prediction and equalization as well as coding strategies and signal design (see [6] and the references therein), to our knowledge the issue of signal activity detection has not been approached yet in this context.

We adopt a Generalized Likelihood Ratio (GLR) approach [7] in order to cope with nuisance parameters in the model (signal and noise powers). The proposed detector is thus a generalization to the time-varying channel case of the Matched Filter (MF) detector [3]. The distribution of the test statistic under both hypotheses is obtained analytically.

After stating the problem in Sec. 2, the GLR test is developed in Sec. 3. Performance is analyzed in Sec. 4, with results given in Sec. 5. Sec. 6 presents the conclusions.

## 2. SYSTEM MODEL

A known length- $N$  signal  $\mathbf{x} \in \mathbb{C}^N$  is to be detected after propagating through a frequency-flat, time-varying channel  $\mathbf{h} \in \mathbb{C}^N$ , and in the presence of additive noise. Thus, when the signal is present, the observation  $\mathbf{y} \in \mathbb{C}^N$  is given by

$$\mathbf{y} = \alpha \mathbf{X} \mathbf{h} + \sigma \mathbf{w}, \quad (1)$$

where  $\alpha$  and  $\sigma$  are unknown constants,  $\mathbf{X} \doteq \text{diag}\{\mathbf{x}\}$ , and the noise vector  $\mathbf{w}$  is zero-mean circularly symmetric complex Gaussian with  $\text{E}\{\mathbf{w}\mathbf{w}^H\} = \mathbf{I}_N$ . The signal  $\mathbf{x}$  is assumed to have unit constant magnitude, so that  $\mathbf{X}^H \mathbf{X} = \mathbf{I}_N$ .

We model the channel time variations using the BEM approach with random coefficients: it is assumed that  $\mathbf{h} = \sqrt{N/K} \mathbf{F} \mathbf{c}$ , where  $\mathbf{F} \in \mathbb{C}^{N \times K}$  has orthonormal columns ( $\mathbf{F}^H \mathbf{F} = \mathbf{I}_K$ ) with  $K \leq N$ , and  $\mathbf{c}$  is zero-mean circularly symmetric complex Gaussian with  $\text{E}\{\mathbf{c}\mathbf{c}^H\} = \mathbf{I}_K$ . (Of particular relevance is the case in which the columns of  $\mathbf{F}$  are given by the first  $K$  vectors in the Fourier basis [4, 8], so that  $K/N$  is proportional to the maximum Doppler shift). Thus, the detection problem can be stated as

$$\mathcal{H}_0 : \mathbf{y} = \sigma \mathbf{w}, \quad \mathcal{H}_1 : \mathbf{y} = \alpha \sqrt{\frac{N}{K}} \mathbf{X} \mathbf{F} \mathbf{c} + \sigma \mathbf{w}. \quad (2)$$

Note that the factor  $\sqrt{N/K}$  in the signal term allows to write the overall power under  $\mathcal{H}_1$  as  $\text{E}\{\|\mathbf{y}\|^2\} = N(\alpha^2 + \sigma^2)$ .

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It will be convenient to introduce the "fractional bandwidth" parameter

$$b \doteq \frac{K}{N} \leq 1. \quad (3)$$

### 3. DERIVATION OF THE GLR TEST

The GLR test [7] is a generalization of the Neyman-Pearson test where nuisance parameters are replaced by their maximum likelihood (ML) estimates under the corresponding hypothesis. For the problem at hand, the GLR test becomes

$$L(\mathbf{y}) \doteq \frac{\max_{\alpha^2 \geq 0, \sigma^2 \geq 0} p(\mathbf{y}; \alpha^2, \sigma^2 | \mathcal{H}_1)}{\max_{\sigma^2 \geq 0} p(\mathbf{y}; \sigma^2 | \mathcal{H}_0)} \underset{\mathcal{H}_0}{\underset{\mathcal{H}_1}{\geq}} \gamma. \quad (4)$$

Under  $\mathcal{H}_0$ , the probability density function (pdf) of  $\mathbf{y}$  is

$$p(\mathbf{y}; \sigma^2 | \mathcal{H}_0) = \frac{1}{(\pi\sigma^2)^N} \exp \left\{ -\frac{1}{\sigma^2} \|\mathbf{y}\|^2 \right\}, \quad (5)$$

and the ML estimate of  $\sigma^2$  under  $\mathcal{H}_0$  is readily seen to be  $\hat{\sigma}_0^2 = \frac{1}{N} \|\mathbf{y}\|^2$ .

On the other hand, under  $\mathcal{H}_1$ , the pdf of  $\mathbf{y}$  is given by

$$p(\mathbf{y}; \alpha^2, \sigma^2 | \mathcal{H}_1) = \frac{1}{\pi^N \det \mathbf{C}} \exp \{ -\mathbf{y}^H \mathbf{C}^{-1} \mathbf{y} \}, \quad (6)$$

where the covariance matrix<sup>1</sup> is given by

$$\mathbf{C} = \alpha^2 \frac{N}{K} \mathbf{X} \mathbf{F} \mathbf{F}^H \mathbf{X}^H + \sigma^2 \mathbf{I}_N. \quad (7)$$

Note that maximizing (6) w.r.t.  $\alpha^2, \sigma^2$  amounts to minimizing

$$\mathbf{y}^H \mathbf{C}^{-1} \mathbf{y} + \log \det \mathbf{C}, \quad (8)$$

and the natural approach is to diagonalize  $\mathbf{C}$ . Let  $\mathbf{G} \in \mathbb{C}^{N \times (N-K)}$  have orthonormal columns and  $\mathbf{F}^H \mathbf{G} = \mathbf{0}$ , so that  $\mathbf{W} \doteq [\mathbf{F} \mathbf{G}]$  is unitary and  $\mathbf{F} = \mathbf{W} [\mathbf{I}_K \mathbf{0}]^T$ . Hence  $\mathbf{X} \mathbf{W}$  is also unitary, and it is clear from (7) that  $\mathbf{C} = \mathbf{X} \mathbf{W} \mathbf{D} \mathbf{W}^H \mathbf{X}^H$  constitutes an eigenvalue decomposition of  $\mathbf{C}$ , where

$$\mathbf{D} \doteq \begin{bmatrix} (\frac{N}{K} \alpha^2 + \sigma^2) \mathbf{I}_K & \mathbf{0} \\ \mathbf{0} & \sigma^2 \mathbf{I}_{N-K} \end{bmatrix}. \quad (9)$$

Therefore, with  $\mathbf{z} \doteq \mathbf{W}^H \mathbf{X}^H \mathbf{y} = [z_0 \ z_1 \ \dots \ z_{N-1}]^T$ , minimizing (8) amounts to minimizing  $\mathbf{z}^H \mathbf{D}^{-1} \mathbf{z} + \log \det \mathbf{D}$ . Let us introduce the quantities

$$\hat{p} \doteq \frac{1}{K} \sum_{i=0}^{K-1} |z_i|^2 \quad \text{and} \quad \hat{q} \doteq \frac{1}{N-K} \sum_{i=K}^{N-1} |z_i|^2 \quad (10)$$

<sup>1</sup>The dependence of  $\mathbf{C}$  with  $\alpha^2$  and  $\sigma^2$  is not explicitly written so as not to overburden the notation, but must be kept in mind.

which are the estimated average powers in the signal and noise-only subspaces, respectively. Then

$$\mathbf{z}^H \mathbf{D}^{-1} \mathbf{z} + \log \det \mathbf{D} = (N-K) \left[ \frac{\hat{q}}{\sigma^2} + \log \sigma^2 \right] + K \left[ \frac{\hat{p}}{\frac{N}{K} \alpha^2 + \sigma^2} + \log \left( \frac{N}{K} \alpha^2 + \sigma^2 \right) \right]. \quad (11)$$

The minimizer of (11), subject to  $\alpha^2 \geq 0$  and  $\sigma^2 \geq 0$ , is

$$(\hat{\alpha}_1^2, \hat{\sigma}_1^2) = \begin{cases} (0, b\hat{p} + (1-b)\hat{q}) & \text{if } \hat{q} \geq \hat{p}, \\ (b(\hat{p}-\hat{q}), \hat{q}) & \text{if } \hat{q} < \hat{p}. \end{cases} \quad (12)$$

(The proof is omitted due to lack of space). Evaluating  $L(\mathbf{y})$  in (4) with the expressions obtained for  $\hat{\sigma}_0^2, \hat{\alpha}_1^2$  and  $\hat{\sigma}_1^2$  and subsequently rearranging terms yields

$$\log L(\mathbf{y}) = \begin{cases} N \cdot \log \frac{\text{A.M.}}{\text{G.M.}}, & \text{if } \hat{p} > \hat{q}, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

where A.M. and G.M. stand for "arithmetic mean" and "geometric mean" respectively, and are given by

$$\text{A.M.} \doteq b\hat{p} + (1-b)\hat{q}, \quad (14)$$

$$\text{G.M.} \doteq \hat{p}^b \hat{q}^{1-b}. \quad (15)$$

It is readily checked that  $N \log \frac{\text{A.M.}}{\text{G.M.}}$  depends on the data only through the ratio  $\hat{p}/\hat{q}$ , and moreover, it is a monotonically nondecreasing function of  $\hat{p}/\hat{q}$  in the range  $\hat{p}/\hat{q} \geq 1$ . Hence, for a given threshold  $\gamma > 1$  in (4), there exists another threshold  $\gamma' > 1$  such that the test (4) is equivalent to  $\hat{p}/\hat{q} \underset{\mathcal{H}_0}{\underset{\mathcal{H}_1}{\geq}} \gamma'$ .

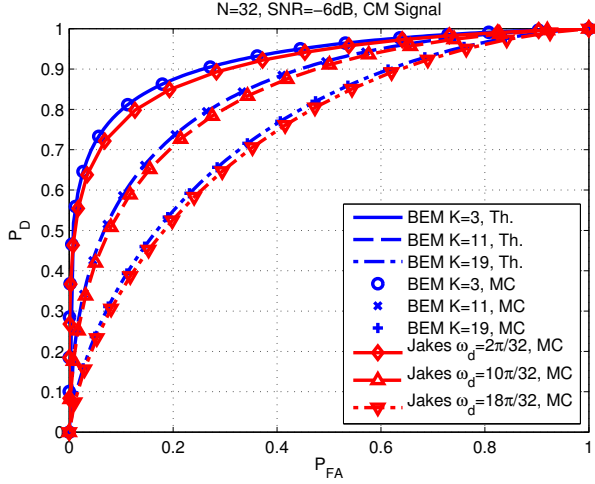
Observe that this test is intuitively satisfying. The first step in the detection process is to correlate the observation with the template signal, obtaining the vector  $\mathbf{X}^H \mathbf{y}$ . This does not change the statistics of the noise component and 'wipes out' the modulation in the signal term. Then the representation of this vector in the orthonormal basis  $\mathbf{W}$  is obtained as  $\mathbf{z} = \mathbf{W}^H (\mathbf{X}^H \mathbf{y})$ . Note that if the Fourier basis is chosen for the BEM, then  $\mathbf{z}$  is the Discrete Fourier Transform of  $\mathbf{X}^H \mathbf{y}$ , and  $\hat{p}, \hat{q}$  are the average values of the periodogram over the "signal bins" and "noise bins", respectively. The GLR test then compares these two values.

An equivalent description of the test is obtained by noting that  $b\hat{p} + (1-b)\hat{q} = \frac{\|\mathbf{z}\|^2}{N} = \frac{\|\mathbf{y}\|^2}{N}$ , from which

$$\frac{\hat{p}}{\hat{q}} = \frac{1-b}{\left( \frac{N\hat{p}}{\|\mathbf{y}\|^2} \right)^{-1} - b}, \quad (16)$$

which is a monotonically nondecreasing function of  $N\hat{p}/\|\mathbf{y}\|^2$ . Thus the GLR test can be recast as  $N\hat{p}/\|\mathbf{y}\|^2 \underset{\mathcal{H}_0}{\underset{\mathcal{H}_1}{\geq}} \gamma''$ .

Also note that the time-invariant case is recovered if  $K = 1$  and  $\mathbf{F} = \frac{1}{\sqrt{N}} \mathbf{1}$ , with  $\mathbf{1}$  the vector of all ones. In that case, since  $z_0 = \frac{1}{\sqrt{N}} \mathbf{1}^T \mathbf{X}^H \mathbf{y} = \frac{1}{\sqrt{N}} \mathbf{x}^H \mathbf{y}$ , the detector reduces to  $|\mathbf{x}^H \mathbf{y}|^2 / \|\mathbf{y}\|^2 \underset{\mathcal{H}_0}{\underset{\mathcal{H}_1}{\geq}} \gamma''$ , which is the Matched Filter detector (normalized by the total observed power in order to account for the lack of knowledge about the noise variance).



**Fig. 1.** Comparison between theoretical and Monte Carlo results for several BEM and Jakes' channels.

#### 4. PERFORMANCE ANALYSIS

Since  $\mathbf{z} = \mathbf{W}^H \mathbf{X}^H \mathbf{y}$  is zero-mean Gaussian with diagonal covariance  $\mathbf{D}$  given by (9),  $\hat{p}$  and  $\hat{q}$  are the sum of squared magnitudes of complex zero-mean i.i.d. Gaussian random variables. Thus, it can be readily checked that:

$$\begin{aligned} k_0 \hat{p} &\sim \chi^2(2K) \text{ under } \mathcal{H}_0, \text{ where } k_0 = 2/\sigma^2, \\ k_1 \hat{p} &\sim \chi^2(2K) \text{ under } \mathcal{H}_1, \text{ where } k_1 = 2/(\frac{N}{K}\alpha^2 + \sigma^2), \\ k_0 \hat{q} &\sim \chi^2(2(N-K)) \text{ under } \mathcal{H}_0 \text{ and } \mathcal{H}_1, \end{aligned}$$

where  $\chi^2(d)$  denotes a chi-square distribution with  $d$  degrees of freedom. The GLR test statistic  $\hat{p}/\hat{q}$  is, therefore, the quotient of two independent chi-square random variables, which is  $F$ -distributed [9]. Hence, under  $\mathcal{H}_0$ ,

$$\frac{k_0 \hat{p}/(2K)}{k_0 \hat{q}/(2(N-K))} = \frac{N-K}{K} \frac{\hat{p}}{\hat{q}} \sim F(2K, 2(N-K)),$$

and, under  $\mathcal{H}_1$ ,

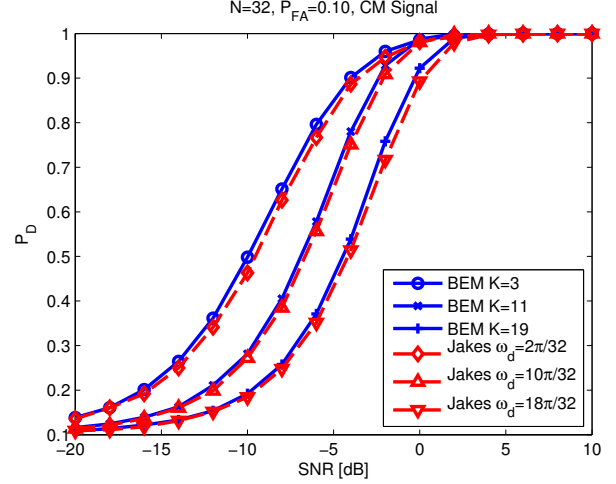
$$\frac{k_1 \hat{p}/(2K)}{k_0 \hat{q}/(2(N-K))} = \frac{k_1}{k_0} \frac{N-K}{K} \frac{\hat{p}}{\hat{q}} \sim F(2K, 2(N-K)).$$

Let  $\rho \doteq \frac{\alpha^2}{\sigma^2}$  denote the SNR. Then one has

$$\frac{k_1}{k_0} = \frac{b}{\rho + b}. \quad (17)$$

Denoting by  $\mathcal{F}_{K,N}(x)$  the cumulative distribution function (cdf) of a  $F(2K, 2(N-K))$ -distributed random variable, the probabilities of detection and false alarm can be written as

$$P_D = 1 - \mathcal{F}_{K,N}\left(\frac{1-b}{\rho+b}\gamma'\right), \quad P_{FA} = 1 - \mathcal{F}_{K,N}\left(\frac{1-b}{b}\gamma'\right). \quad (18)$$



**Fig. 2.** Probability of detection vs SNR for several BEM and Jakes' channels. The Doppler spread of the channel is known.

Eliminating the threshold  $\gamma'$  from (18) yields

$$P_D = 1 - \mathcal{F}_{K,N}\left(\frac{b}{\rho+b} \cdot \mathcal{F}_{K,N}^{-1}(1 - P_{FA})\right). \quad (19)$$

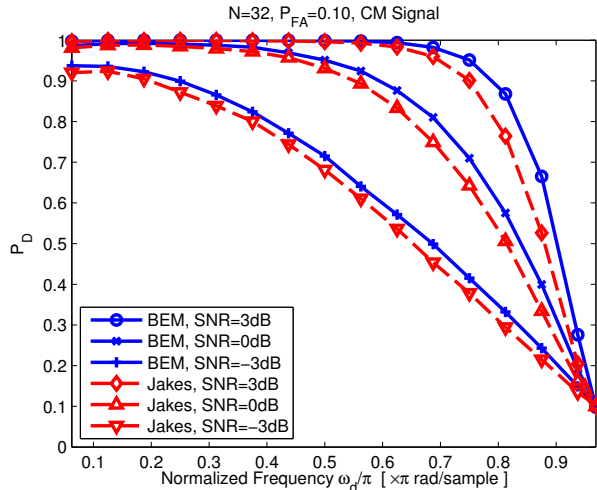
#### 5. SIMULATION RESULTS

We illustrate the performance of the GLR test for two kinds of flat-fading Rayleigh channels: namely, the BEM model described in the previous sections, and the dense scatterer (Jakes) model [6, 10]. The basis functions used to generate the BEM channel samples are the  $K$  elements in the Fourier basis corresponding to the low frequency region, chosen symmetrically around DC.  $K$  is taken odd, so that the Doppler spread is  $\omega_d = \frac{K-1}{2} \frac{2\pi}{N}$ . On the other hand, under Jakes' model, the channel covariance matrix  $\mathbf{E}\{\mathbf{h}\mathbf{h}^H\}$  is Toeplitz with the element  $(k, l)$  given by  $J_0(\omega_d(k-l))$ , where again  $\omega_d$  denotes the maximum Doppler frequency (Doppler spread).

The GLR test considered is based on the Fourier basis. Hence, if  $\omega_d$  is assumed known, a reasonable choice for  $K$  satisfies  $\omega_d = \frac{K-1}{2} \frac{2\pi}{N}$ . Thus, unless otherwise stated, the detector uses the value  $K = 1 + \lceil \frac{N\omega_d}{\pi} \rceil$ .

Fig. 1 shows the Receiver Operating Characteristic (ROC) of the detector for both channels, for several Doppler spreads. In the BEM case, the observations fit the model used to develop the test, and hence simulation results perfectly agree with the analytical result (19). As expected, the detection performance degrades for Jakes' model, although this degradation is small. This is also seen in Fig. 2, which shows the probability of detection ( $P_D$ ) vs SNR. The SNR penalty incurred with Jakes' channels is in the order of 0.5 dB for this example.

The impact of the Doppler spread on the performance is shown in Fig. 3. As it can be seen,  $P_D$  worsens as  $\omega_d$  in-



**Fig. 3.** Probability of detection vs the (known) Doppler spread for several SNR values.

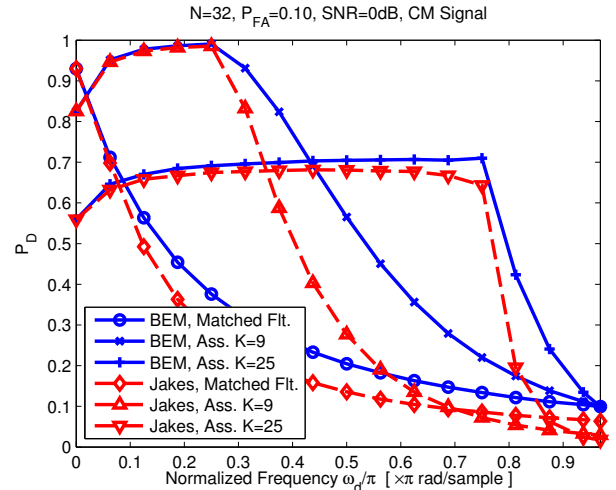
creases, even when the latter is perfectly known. Note that for  $\omega_d = \pi$  (the largest Doppler spread possible) one has  $P_D = P_{FA}$  and the detector becomes useless. This is because in that case the distributions of  $\mathbf{y}$  under both hypotheses coincide.

Finally, we consider the case in which the actual value of the channel Doppler spread is not available to the detector. Fig. 4 shows  $P_D$  vs the actual Doppler spread for several values of  $K$  used in the detection. Although the Matched Filter detector (i.e., the GLR detector with  $K = 1$ ) is the best choice for time-invariant channels, its performance quickly degrades as soon as fading is introduced. It is clear from Fig. 4 that overestimating the true Doppler spread is less detrimental than underestimating it.

## 6. CONCLUSIONS

A novel detector for known constant magnitude signals in frequency-flat fast fading channels was presented. The test is robust to uncertainties in the noise variance. Channel time variations are dealt with by resorting to a Basis Expansion Model; the selection of the number of elements in the basis can be done if some rough estimate of the channel Doppler spread is available. The distributions of the test statistic under both hypotheses were developed in closed form. As expected, faster time variations introduce a degradation in detection performance, but the proposed methods are much more robust in that sense than the standard Matched Filter detector, which was designed for time-invariant channels.

The coefficients used in the basis expansion of the channel were modeled as i.i.d. Gaussian. It is of interest to explore other options, for example to use a deterministic but unknown model for the channel. Future work should also address the case of unknown and/or nonconstant magnitude signals.



**Fig. 4.** Probability of detection vs the (unknown) Doppler spread for different assumed values of  $K$ .

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