ANALYSIS AND DESIGN OF MULTIRATE SYNCHRONOUS SAMPLING SCHEMES FOR SPARSE MULTIBAND SIGNALS

M. E. Domínguez-Jiménez

E.T.S.I. Industriales, UPM
28006 Madrid, Spain
elena.dominguez@upm.es

N. González-Prelcic

E.E. Telecomunicación, Universidade de Vigo
36313 Vigo, Spain
nuria@gts.uvigo.es

ABSTRACT

We consider the problem of developing efficient sampling schemes for multiband sparse signals. Previous results on multicoset sampling implementations that lead to universal sampling patterns (which guarantee perfect reconstruction), are based on a set of appropriate interleaved analog to digital converters, all of them operating at the same sampling frequency. In this paper we propose an alternative multirate synchronous implementation of multicoset codes, that is, all the analog to digital converters in the sampling scheme operate at different sampling frequencies, without need of introducing any delay. The interleaving is achieved through the usage of different rates, whose sum is significantly lower than the Nyquist rate of the multiband signal. To obtain universal patterns the sampling matrix is formulated and analyzed. Appropriate choices of the parameters, that is the block length and the sampling rates, are also proposed.

Index Terms— Multicoset sampling, multirate sampling, compressive sampling, multiband sparse signal, universal pattern, Kruskal rank.

1. INTRODUCTION

Time-Interleaved Analog-to-Digital Converters (TI-ADCs) have been proposed for the implementation of universal multicoset sampling patterns, which guarantee the recovery of a multiband sparse signal (i.e., only a small number of frequency subbands are occupied) from a small number of samples. Figure 1 shows the block diagram of this architecture, where $f_s = 1/T$ denotes the Nyquist rate, p the number of branches, τ_i are different delays and L is the length of the block signal that has to be reconstructed. The locations of the K active subbands are not known *a priori.*, but making usage of the sparsity of the signal and appropriately selecting the delays and the block length, the multiband signal can be reconstructed if $p \geq K$ [1, 2, 3].

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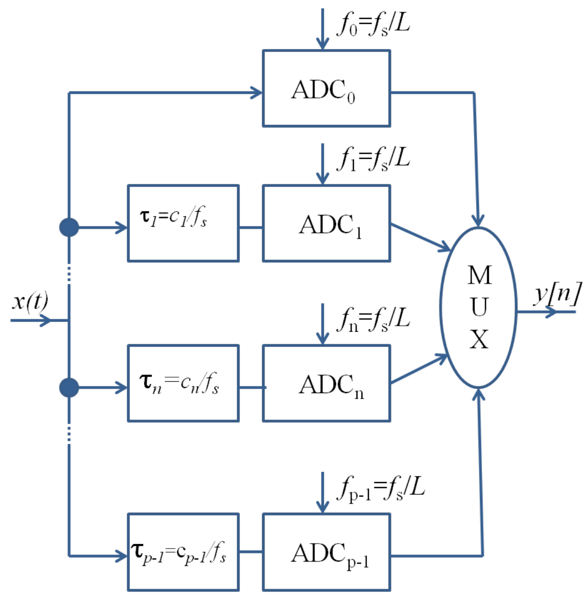

Fig. 1. Block diagram of the most commonly used TI-ADC system.

The scheme proposed in [2] can also be implemented with a TI-ADC architecture similar to that of Fig. 1, with a random selection of the delays. The main drawback of random selection approaches is that they require to have L ADCs constantly working. In contrast, fixed channel selection methods allow to reduce the number of branches of the hardware system p, to the number of active bands K, with the corresponding reduction in area size and power consumption.

Thus, in [1], a fixed branch selection is assumed. The uniform sampling grid (at Nyquist rate) is divided into blocks of L consecutive samples, and then only p out of these L samples are acquired. This can be implemented with the p branches of the TI-ADC system in Fig. 1, using $\tau_i = c_i T$ with integers $0 \leq c_0 < c_1 < \cdots < c_{p-1} \leq L-1$ and equal sampling frequencies $f_i = f_s/L$. Only certain selections of the p output channels lead to the reconstruction of the original signal, although the authors of [1] do not provide a design criterion for this; in [4], it is proven that any sampling pattern is universal if L is prime, but the issue of finding an appropriate set $\{c_k\}_{k=0}^{p-1}$ (a *multicoset code*) for any given value of L is stated to be a combinatorial problem.

However, in [3], the matrix involved in the corresponding reconstruction equation is deeply analyzed, which makes possible to define a method for the design of universal multicoset sampling patterns, that is, patterns which guarantee perfect reconstruction of the sparse multiband signal within the framework of compressive sampling. Thus, p is selected from the number of active bands, and delays are obtained as consecutive numbers from an arithmetic progression of difference d coprime with the block length L. Difference d can be either positive or negative, and smaller or greater than L. This is the first systematic method proposed for this goal; previous multicoset codes were either obtained by direct search [5] or relied on the choice of a prime value of L [4].

In this paper we propose and analyze an alternative and more flexible architecture to generate universal patterns: a multirate synchronous sampling scheme with M branches, as depicted in Figure 2. In this case we have a set of TI-ADCs working synchronously and using different sampling frequencies for each ADC (with the sum of of these different sampling rates lower than the Nyquist rate); the design problem becomes the search for the sampling frequencies and the block length that lead to universal sampling patterns. A sampling scheme based on this architecture was first proposed in [6], but the focus was on the development of a reconstruction scheme for a set of different and arbitrarily chosen sampling frequencies; no criterion was proposed on how to choose the sampling frequencies. In [7, 8, 9] a multirate synchronous sampling scheme with coprime subsampling factors was also proposed, but not with a reconstruction objective but for the estimation of the autocorrelation function of a wide-sense stationary signal. Finally, a multirate sampling scheme was also proposed in [10], but in this case delays between ADCs were assumed, and the work was also focused on the reconstruction procedure, not in the design of the sampling scheme.

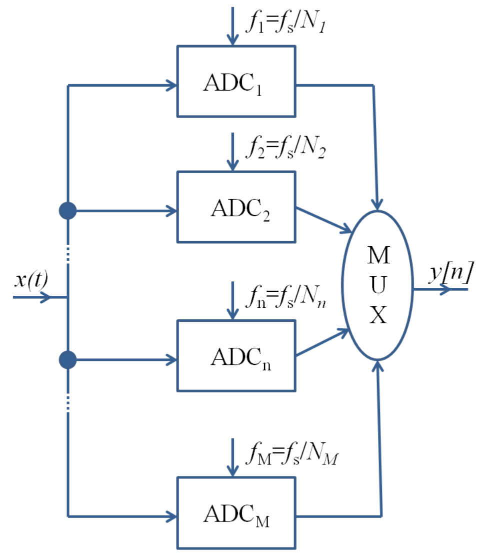

Fig. 2. Block diagram of a synchronous multirate system.

2. PROBLEM STATEMENT

As in [1], we assume a complex-valued $K-$sparse multiband signal (the number of active bands is K) $x(t)$, bandlimited to $[0, f_s]$. The upper bound B for the bandwidth of these bands is known. The sampling stage is implemented as in Fig. 2. Thus, for each block of L Nyquist-rate samples, p of them are acquired, with indices $0 \leq c_0 < c_1 < \cdots < c_{p-1} \leq L-1$.

Following [1], we define the sampling pattern $C = \{c_k\}_{k=0}^{p-1}$. The reconstruction of the multiband signal from the acquired samples requires the selection of L, p and C such that $X(f)$ can be reconstructed based on

$$\boldsymbol{y}(f) = \frac{1}{LT}\boldsymbol{A}\boldsymbol{x}(f),$$

with

$$\begin{aligned} y_k(f) &= X_{c_k}\left(e^{j2\pi fT}\right), \quad k=0,\dots,p-1 \\ x_l(f) &= X\left(f+\frac{l}{LT}\right), \quad l=0,\dots,L-1 \end{aligned}$$

and where $\boldsymbol{A}$ is a $p \times L$ Vandermonde matrix with elements

$$a_{k,l} = \exp\left(j\frac{2\pi}{L}c_k l\right), \; k=0,...,p-1, \; l=0,...,L-1. \quad (1)$$

It is noted in [1] that $\boldsymbol{x}(f)$ can be recovered from $\boldsymbol{y}(f)$ if $L \leq f_s/B$, $p \geq K$ and $\boldsymbol{A}$ has Kruskal-rank[1] equal to p. For given L and p, a sampling pattern C that results in a fully Kruskal-rank $\boldsymbol{A}$ is termed *universal* [1].

In this work we will consider the case where these indices are selected from the sampling positions provided by a set of different uniform sampling sequences $x_i[n]$ at rates f_s/N_i, $(i = 0 \ldots M-1)$: In other words, this means considering indices of the form nN_i, $i = 0, \cdots, M-1$. Note that not all the samples generated by the set of ADCs have to be used for the reconstruction of the original signal. Our goal will be obtaining the sampling frequencies, i.e. the subsampling factors N_i and the block length that lead to a fully Kruskal-rank matrix.

[1]The Kruskal- or K-rank of a matrix is the largest value of m such that every subset of m columns of the matrix is linearly independent.

3. ANALYSIS OF THE RECONSTRUCTION MATRIX FOR A SYNCHRONOUS MULTIRATE SAMPLING SCHEME

3.1. The simplest case: only one ADC

In this case the sampling pattern is simply obtained by extracting 1 out of N_1 samples in each block of length $L = pN_1$. This is an obvious selection for L, since in this way, we obtain p different samples from each block $[0, L-1]$:

$$c_k = kN_1, \quad k = 0, ..., p-1. \tag{2}$$

Let us study the K-rank of the corresponding matrix $\boldsymbol{A}$. Our first result guarantees that $\boldsymbol{A}$ has not full K-rank (p); on the contrary, its K-rank is always minimum:

Theorem 1. *Let $L = pN_1$ and let us define p samples in $[0, L-1]$ as*

$$c_k = kN_1 \quad k = 0, ..., p-1.$$

Then the corresponding $p \times L$ Vandermonde matrix $\boldsymbol{A}$ has K-rank equal to 1.

Proof. Let us denote $w = \exp\left(\frac{2\pi}{L}j\right)$ the primitive $L-$root of 1. The $(k+1)$-th row of $\boldsymbol{A}$ contains the powers of the complex number $w_k = \exp\left(\frac{2\pi}{L}jc_k\right)$. In this case this number is

$$w_k = \exp\left(\frac{2\pi}{L}jkN_1\right) = \exp\left(\frac{2\pi}{p}jk\right).$$

Note that

$$w_k^p = 1 = w_k^0.$$

This implies that $\boldsymbol{A}$ has two equal columns: the first one (whose elements are $w_k^0 = 1$) and the $(p+1)$-th column (whose elements are $w_k^p = 1$). Therefore the K-rank of $\boldsymbol{A}$ is $1 < p$. □

Remark: This result asserts that there is no universal sampling pattern of the kind (2) if the block size is $L = pN_1$. When L is multiple of p, only 1 active subband can be reconstructed.

In the next Section we will study the existence of universal sampling patterns for a multirate synchronous system with only two branches and the selection for the block length $L = N_1 N_2$.

3.2. The two ADCs case

We will now consider another approach, with $L = N_1 N_2$. Unlike the last case ($L = pN_1$, where the p samples were obtained as the output of 1 subsampler), in this case we will consider the output of 2 converters: one which extracts the samples of the kind kN_2, $k = 0, \cdots, N_1 - 1$, and another one which provides the samples $mN_1 \quad m = 1, ..., N_2 - 1$. The selection $L = N_1 N_2$ coincides with the natural repetition cycle of the indices genrated by two converters working synchronously at different rates.

We will assume that N_1, N_2 are coprime; this way, we have a total amount of $p = N_1 + N_2 - 1$ different samples (if N_1, N_2 are not coprime, there are coincident samples: $kN_2 = mN_1$ for some $k < N_1$ and $m < N_2$). Under this assumption, we provide the next result on the K-rank of the Vandermonde matrix $\boldsymbol{A}$:

Theorem 2. *Let $L = N_1 N_2$ with $2 \leq N_1, N_2$ coprime numbers. Let us consider the $N_1 + N_2 - 1$ different samples*

$$c_k = kN_2 \quad k = 0, ..., N_1 - 1 \tag{3}$$

$$c_{m+N_1} = mN_1 \quad m = 0, ..., N_2 - 2. \tag{4}$$

If we choose a subset $\{c_i\}$ of p samples extracted from these ones, and $p \geq 4$, then the K-rank of $\boldsymbol{A}$ is not maximum (p).

In particular, if we consider all these $N_1 + N_2 - 1$ samples, then the K-rank of $\boldsymbol{A}$ is not maximum (p).

Proof. The complex numbers

$$w_k = \exp\left(\frac{2\pi}{L}jc_k\right), \quad k = 0, ..., N_1 + N_2 - 2$$

are roots of 1. In fact, for $k = 0, ..., N_1 - 1$, they are N_1-roots of 1, because

$$w_k = \exp\left(\frac{2\pi}{L}jkN_2\right) = \exp\left(\frac{2\pi}{N_1}jk\right) \Rightarrow w_k^{N_1} = 1.$$

And for $k = N_1, \cdots, N_1 + N_2 - 2$, they are N_2-roots of 1

$$w_{m+N_2} = \exp\left(\frac{2\pi}{L}jmN_1\right) = \exp\left(\frac{2\pi}{N_2}jm\right) \Rightarrow w_k^{N_2} = 1.$$

This means that all w_k are roots of the polynomial

$$Q(z) = \left(z^{N_1} - 1\right)\left(z^{N_2} - 1\right) = z^{N_1+N_2} - z^{N_1} - z^{N_2} + 1.$$

In other words, for any $k = 0, ..., N_1 + N_2 - 2$,

$$w_k^{N_1+N_2} - w_k^{N_1} - w_k^{N_2} + w_k^0 = 0.$$

From a matricial point of view, this means that there are 4 columns of $\boldsymbol{A}$ which are linearly dependent. In fact, recall from Equation (1) that the $(m+1)$-th column of $\boldsymbol{A}$ contains the $m-$th powers of w_k, w_k^m; hence, the last equation assures that the $(N_1 + N_2 + 1)$-th column of $\boldsymbol{A}$, its $(N_1 + 1)$-th column, its $(N_2 + 1)$-th column and its first column are linearly dependent. Note that the index column $N_1 + N_2 + 1$ is smaller than or equal to L because $N_1 + N_2 < 2\max(N_1, N_2) \leq L$.

The existence of 4 columns of $\boldsymbol{A}$ which are linearly dependent implies that the K-rank is at most 3. Then if we extract a number of samples $p \geq 4$ then the K-rank will not be maximum. Moreover, if we choose all the samples, the K-rank will not be maximum either: as $p = N_1 + N_2 - 1$ then $p \geq 4$ since $p = N_1 + N_2 - 1 > 2N_1 - 1 \geq 3$. □

Example: If $L = 15 = 3{\cdot}5$, we generate $3 + 5 - 1 = 7$ different samples: $(0, 3, 6, 9, 12)$ and $(7, 14)$. But the previous proof shows that any of those samples provides a root of the polynomial

$$Q(z) = \left(z^5 - 1\right)\left(z^3 - 1\right) = z^8 - z^5 - z^3 + 1$$

which means that the 7×15 matrix $\boldsymbol{A}$ has 4 linearly dependent columns: the ones indexed by $0, 3, 5, 8$. This means that the K-rank at maximum is 3. The same holds if we extract $p \geq 4$ out of those 7 samples: the K-rank of the corresponding $p \times 15$ matrix will be at most 3.

Remark: We have proven that the synchronous sampling for $L = N_1 N_2$ (N_1, N_2 coprime) yields $N_1 + N_2 - 1$ different samples, but if we extract from these a pattern $\{c_k\}$ of more than 3 samples, the K-rank will never be maximum. Thus, the unique solution for the synchronous sampling is just to consider p =1,2 or 3 samples. But notice that, even for that case, nothing assures that the maximum K-rank (p) will be reached.

It would be nice to guarantee that, by taking just $p = 3$ samples we would obtain K-rank equal to 3. But the following result assures that it is not possible either, if $N_1, N_2 > 2$:

Theorem 3. *Let $L = N_1 N_2$ with N_1, N_2 coprime.*

1. *If we just extract $p = 3$ different samples (say, 0, kN_1 and mN_2) then the K-rank is not maximum (3)*
2. *The K-rank is not maximum either, when we extract 3 samples of the kind kN_1, mN_2 and $m'N_2$ (being $N_1 < N_2$).*
3. *If $N_1, N_2 > 2$, then the K-rank is not maximum either, when we extract 3 samples of the kind kN_1, $k'N_1$ and mN_2.*

Proof.

1. Let us study the case of $p = 3$ different samples: 0, kN_1 and mN_2. The matrix $\boldsymbol{A}$ has dimensions $3 \times L$. We now show that not any 3×3 submatrix of order 3 of $\boldsymbol{A}$ is invertible: in fact, if $N_1 = \min(N_1, N_2)$ then the submatrix built with the 3 columns indexed by 0, N_1, $2N_1$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & w^{kN_1^2} & w^{2kN_1^2} \\ 1 & w^{mN_2N_1} & w^{2mN_2N_1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w^{kN_1^2} & w^{2kN_1^2} \\ 1 & 1 & 1 \end{pmatrix}$$

which obviously does not have rank 3. We have only used that $w^{N_1N_2} = w^L = 1$ and the fact that $2N_1 < L$ which is true because $N_2 > N_1 \geq 2$ so $L = N_2N_1 > 2N_1$.

2. If $N_1 = \min(N_1, N_2)$, the same happens if we choose different samples kN_1, mN_2 and $m'N_2$: the same submatrix, built with the 3 columns indexed by $0, N_1, 2N_1$, has not maximum rank either:

$$\begin{pmatrix} 1 & w^{kN_1^2} & w^{2kN_1^2} \\ 1 & w^{mN_2N_1} & w^{2mN_2N_1} \\ 1 & w^{m'N_2N_1} & w^{2m'N_2N_1} \end{pmatrix} = \begin{pmatrix} 1 & w^{kN_1^2} & w^{2kN_1^2} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

3. Let us finally consider the case kN_1, $k'N_1$ and mN_2: the columns indexed by $0, N_2$ and $2N_2$ would give a 3×3 matrix with the first and second rows filled by 1's (hence, not invertible):

$$\begin{pmatrix} 1 & w^{kN_1N_2} & w^{2kN_1N_2} \\ 1 & w^{k'N_1N_2} & w^{2k'N_1N_2} \\ 1 & w^{mN_2^2} & w^{2mN_2^2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & w^{mN_2^2} & w^{2mN_2^2} \end{pmatrix}.$$

Note that in this case the index of the last column, $2N_2$ is smaller than L because $2N_2 < N_1N_2 = L$.

□

Hence for $L = N_1N_2$ with coprime $N_1, N_2 > 2$ the K-rank is at maximum 2.

Finally let us consider the case $L = 2N_2 = 2N$ with N odd. Fortunately, the following result assures that it is possible to reach maximum K-rank with $p = 3$ samples:

Theorem 4. *Let $L = 2N$ with N an odd prime, and let us consider an odd number $1 \leq m < N$. Then the sampling pattern $N - m, N, N + m$, is universal, since it yields maximum K-rank (3).*

Proof. Any 3×3 submatrix of $\boldsymbol{A}$ is of the form

$$\begin{pmatrix} w^{(N-m)k} & w^{(N-m)k'} & w^{(N-m)k''} \\ w^{Nk} & w^{Nk'} & w^{Nk''} \\ w^{(N+m)k} & w^{(N+m)k'} & w^{(N+m)k''} \end{pmatrix}.$$

We will prove that this submatrix is invertible for any $0 \leq k < k' < k'' \leq L - 1$. By extracting the factors

$$w^{(N-m)k}, w^{(N-m)k'}, w^{(N-m)k''}$$

respectively, from the first, second and third column, its rank is the same as the rank of

$$\begin{pmatrix} 1 & 1 & 1 \\ w^{mk} & w^{mk'} & w^{mk''} \\ w^{2mk} & w^{2k'} & w^{2mk''} \end{pmatrix}.$$

The latter matrix is a Vandermonde matrix of the numbers $w^{mk}, w^{mk'}, w^{mk''}$. It is invertible if and only if those 3 numbers are different. But if it happened to be $w^{mk} = w^{mk'}$, then $mk = mk' \bmod L$, say, $m(k - k') = 2Nr,\ r \in \mathbb{Z}$. As m is odd, we have that

$$m\left((k - k')/2\right) = Nr, \quad r \in \mathbb{Z}.$$

Being N prime, N must be either a divisor of m (which is impossible) or a divisor of $(k - k')/2 < N$. The unique chance is that $(k-k')/2=0$, leading to $k=k'$ which is a contradiction, because $k<k'$ by hypothesis. The contradiction comes from the assumption that the submatrix can be singular. Hence, any submatrix of order 3 is nonsingular, and $\boldsymbol{A}$ has maximum rank (3). □

Examples:

- An easy way to obtain the maximum K-rank is by taking $L=6=2{\cdot}3$. As 3 is prime, we can just extract 3 samples: $3-1, 3, 3+1$ (say, the samples indexed as $2,3,4$). This method is equivalent to applying 2 synchronous samplers in blocks of length $L=6$: a subsampler takes 1 out of 2 samples (providing the samples indexed $0,2,4$) and the other one picks 1 out of 3 samples (providing the ones indexed $0,3$). If each one discards its first sample, we just extract the sampling pattern $2,3,4$. Our contribution here guarantees that those 3 samples are sufficient for reconstructing $p=3$ bands.
- Another example: for $L=10$ we can consider sampling pattern $2,5,8$, or the sampling pattern $4,5,6$. Each one guarantees the reconstruction of 3 subbands.

In summary, we provide an easy way to obtain 3 bands of sparse multiband signals. It suffices to take blocks of length $L=6$, or in the general case, blocks of length $L=2N\geq 6$ (with N prime) and choose 3 samples as stated in Theorem 4. However, no more than 3 subbands can be reconstructed when taking $L=N_1N_2$, as we have already proven.

4. CONCLUSIONS

We have analyzed the reconstruction problem of sparse multiband signals when using a synchronous multirate sampling scheme. For the simplest case of using only one ADC at rate f_s/N_1, if the block length is a multiple of the subsampling factor N_1 we demonstrate that only 1 subband can be recovered. On the other hand, when using two synchronous converters with different sampling rates and the block length is obtained as the product of the two coprime subsampling factors, we conclude that we can reconstruct at most 3 subbands, regardless of using a great number of samples. Moreover, the number of recovered subbands can be even smaller (equal to 2 or 1). In order to overcome this problem, we propose a new selection of the block length that leads to a sampling pattern which guarantees the reconstruction of any 3 subbands. As a further reseacrh, new selections of the block length as a function of the subsampling factors have to be proposed and studied, in order to guarantee that any number of active bands in the multiband signal can be reconstructed from the sub-Nyquist pattern generated with a synchronous multirate sampling scheme.

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