Cost minimization interpretation of the Fourth Power phase estimator and links to the Multimodulus algorithm

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Abstract

It is shown that the fourth power phase estimator minimizes or maximizes (depending on a condition on the transmitted constellation) the cost function associated to the recently proposed multimodulus algorithm (MMA) for blind equalization. Implications for operation of MMA are then discussed.

I. Introduction

In burst-mode digital transmission systems adopting coherent demodulation, phase recovery within each burst becomes a crucial issue. Among the blind (or non-data-aided) carrier phase estimators available from the literature, perhaps the fourth power estimate appears to be the most popular. It is known to yield an approximate maximum likelihood estimator in the limit of small SNR [2]. Serpedin *et al.* have shown in [4] that several seemingly different phase estimators are in fact equivalent to the fourth power estimator.

This letter presents a reinterpretation of the fourth power estimator in terms of the minimization of a dispersion-based cost function. This cost has been recently proposed in the context of blind equalization [3], [5] as the basis of the so-called multimodulus algorithm (MMA). One advantage of MMA over the standard constant modulus algorithm (CMA) [1] is precisely its ability to perform blind phase acquisition. Thus, this inherent phase recovery property of MMA can be seen as a 'built-in' fourth power estimator;

consequently one cannot expect MMA to synchronize the phase with constellations for which the fourth power estimator is not well suited, such as M-PSK ($M > 4$).

II. Problem statement and derivation

In the absence of channel impairments other than phase offset and noise, the received samples can be written as

$$Y(k) = X(k)e^{j\theta} + N(k) = Y_r(k) + jY_i(k), \quad 1 \le k \le K, \tag{1}$$

where $X(k)$ is the complex-valued transmitted symbol, $N(k)$ is complex-valued additive noise, assumed independent of the symbols and circular (i.e. $E\{N^p(k)\} = 0$ for all positive integers p), and θ is the phase angle to be determined from the observed values $\{Y(k)\}$.

Let ϕ be a candidate estimate, and define the derotated samples

$$\hat{X}(k) = Y(k)e^{-i\phi} = \hat{X}_r(k) + j\hat{X}_i(k). \tag{2}$$

The multimodulus cost function [3], [5] is defined as the sum of the dispersions of the real and imaginary parts of the derotated signal:

$$J(\phi) = E\{(\hat{X}_r^2(k) - \gamma_r)^2 + (\hat{X}_i^2(k) - \gamma_i)^2\}, \tag{3}$$

for some constants γ_r, γ_i. We will assume that $\gamma_r = \gamma_i$; in that case, it is readily seen by expanding (3) that this constant merely adds a constant term to J, so that we can take $\gamma_r = \gamma_i = 0$ and $J(\phi) = E\{\hat{X}_r^4(k) + \hat{X}_i^4(k)\}$. By noting that

$$\frac{\partial \hat{X}_r(k)}{\partial \phi} = \hat{X}_i(k), \qquad \frac{\partial \hat{X}_i(k)}{\partial \phi} = -\hat{X}_r(k), \tag{4}$$

the derivative of the cost is seen to be

$$\begin{aligned} \frac{\partial J(\phi)}{\partial \phi} &= 4E\{\hat{X}_r(k)\hat{X}_i(k)[\hat{X}_r^2(k) - \hat{X}_i^2(k)]\} \\ &= \operatorname{Im} E\{\hat{X}^4(k)\}. \end{aligned} \tag{5}$$

After some algebra, this can be written explicitly in terms of ϕ as

$$\begin{aligned} \frac{\partial J(\phi)}{\partial \phi} &= 4E\{Y_r(k)Y_i(k)[Y_r^2(k) - Y_i^2(k)]\}\cos 4\phi \\ &\quad - E\{Y_r^4(k) + Y_i^4(k) - 6Y_r^2(k)Y_i^2(k)\}\sin 4\phi \\ &= \operatorname{Im} E\{Y^4(k)\}\cos 4\phi - \operatorname{Re} E\{Y^4(k)\}\sin 4\phi. \end{aligned} \tag{6}$$

Therefore the stationary points of the cost, at which (6) vanishes, are given by

$$\tan 4\phi = \frac{\operatorname{Im} E\{Y^4(k)\}}{\operatorname{Re} E\{Y^4(k)\}},$$

or equivalently

$$\phi = \frac{1}{4} \operatorname{angle} E\{Y^4(k)\}. \tag{7}$$

Differentiating (5) and using (4), the second derivative of the cost is found:

$$\frac{\partial^2 J(\phi)}{\partial \phi^2} = -4 \operatorname{Re} E\{\hat{X}^4(k)\}. \tag{8}$$

Hence we see that $E\{\hat{X}^4(k)\}$ is real at the stationary points of J, becoming negative at the minima and positive at the maxima.

For circular noise independent of the data, it follows that $E\{\hat{X}^4(k)\} = E\{X^4(k)\}e^{j(\theta-\phi)}$. From (5) and (8), it is seen that the noise does not alter the location or character (minimum/maximum) of the stationary points under these conditions.

If the constellation to which $\{X(k)\}$ belong has quadrant symmetry, and if the symbols are drawn equiprobably, then $E\{X^4(k)\}$ is real. In that case, the stationary points of J are given by

$$\operatorname{Im} E\{\hat{X}^4(k)\} = E\{X^4(k)\} \sin 4(\theta - \phi) = 0 \quad \Rightarrow \quad \phi = \theta + n\pi/4,$$

and

$$\left. \frac{\partial^2 J(\phi)}{\partial \phi^2} \right|_{\phi=\theta+n\pi/4} = (-1)^{n+1} 4E\{X^4(k)\}.$$

Hence consecutive minima (or maxima) of $J(\phi)$ are $\pi/2$ rad apart, which reflects the inherent ambiguity due to the quadrant symmetry of the constellation.

The fourth power estimator is defined as [2], [4]

$$\hat{\theta} = \frac{1}{4} \operatorname{angle} \left[E\{X^{*4}(k)\} E\{Y^4(k)\} \right]. \tag{9}$$

Note that $E\{X^{*4}(k)\} = E\{X^4(k)\}$ since this quantity is real. Then in view of (7) and (8), it is seen that the fourth power estimate minimizes or maximizes the cost J according to the sign of $E\{X^4(k)\}$.

III. Effect in the multimodulus algorithm

As MMA is designed to minimize (3), phase correction (modulo $\pi/2$) will be accomplished provided that $E\{X^4(k)\} < 0$. Most rectangular and cross QAM constellations satisfy this property. Other constellations, such as that of the CCITT V.29 standard, do not, so that if the MMA is employed in those systems the equalizer output should be rotated by $\pi/4$ rad. For QPSK, the constellation $\{(\pm 1 \pm j)/\sqrt{2}\}$ satisfies $E\{X^4(k)\} < 0$, while $\{\pm 1, \pm j\}$ does not. Finally, for M-PSK constellations with $M > 4$, $E\{X^4(k)\} = 0$ (and in fact $J(\phi)$ becomes flat), so that MMA cannot compensate for the phase in these systems.

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