Efficient Real Single-Tone Frequency Estimator Based on a Normalized IIR Notch Filter

Abstract

A novel unbiased frequency estimator for a single real-valued sinusoid in white noise is proposed, which is based on a normalized second-order Infinite Impulse Response (IIR) notch filter, an can be seen as an extension of the Pisarenko Harmonic Decomposer (PHD) estimator. The new estimator inherits the simplicity of the PHD estimator and the high accuracy and selectivity achieved by an IIR filter. As the coefficient determining the pole angle of the IIR notch filter depends on the unknown frequency value, an iterative scheme is proposed in which this coefficient is computed from the frequency estimate from the previous iteration. An analytic expression for the Mean Square Error at each iteration is derived. Simulation results show that this iterative method can approach the Cramer-Rao Lower Bound in a few iterations under different Signal to Noise Ratio levels, provided that the data record length is sufficiently large.

I. Introduction

Estimation of the frequency of a sinusoidal signal in the presence of broad-band noise is a common problem in signal processing [1]. Numerous techniques have been developed for its treatment [1], [2], including Maximum Likelihood (ML) estimation [3], notch filtering [4] and methods based on the linear prediction (LP) property of sinusoidal signals such as Prony's method, instrumental variable methods [5], iterative filtering [6], Yule-Walker methods [7] and subspace schemes such as truncated singular value decomposition, MUSIC and ESPRIT [2].

The ML estimator is known to be statistically efficient in the sense that its variance asymptotically achieves the Cramer-Rao Lower Bound (CRLB), but it requires the maximization of a highly nonlinear and multimodal cost function [3], [8], with the corresponding high computational burden. On the other hand, computationally simple estimators can be obtained by exploiting the LP property of the sinusoidal signal, but they result in suboptimum performance. Among these, the Pisarenko Harmonic Decomposer (PHD) [9] is of historical interest, since it was the first to exploit the eigenstructure of the covariance matrix of the observations, and its performance has been extensively studied [10], [11]. Interestingly, for the case of a real-valued single sinusoid the PHD estimator can be implemented in a very simple way [12].

Some attempts to improve the performance of the PHD estimator can be found in the literature. In [13] a variant termed *Reformed* PHD (RPHD) was proposed. Its performance is superior to that of the original PHD, although the statistical analysis in [13] revealed its inefficiency. Another estimator in the Pisarenko framework was derived in [14] from a generalized eigenvalue problem, as a solution to a constrained Weighted Least Squares criterion. The high computational load of this method, however, precludes its application to situations requiring rapid frequency estimation.

The above estimators are esentially *off-line* (i.e. batch) methods. A different approach to frequency estimation is the use of *on-line* (i.e. adaptive) notch filters. In fact, adaptive versions of Pisarenko's method are possible, which attempt to match the angular positions of the zeros of a Finite Impulse Response (FIR) filter to the unknown frequencies of the sinusoids. However, high noise rejection and sharp cutoff bandpass characteristics are desirable traits which can only be obtained with very high order FIR structures, and thus Infinite Impulse Response (IIR) notch filters have become a popular choice. Several algorithms have been developed for the adaptation of these systems, essentially seeking the minimum point of some cost function; see e.g. [15, Ch. 10] and the references therein.

It is natural then to ask whether off-line frequency estimators based on the idea of IIR notch filtering could be devised. The main difficulty in this sense is the fact that the notch filter output is not a linear function of the IIR filter parameters, and thus no Wiener-Hopf type closed-form expression for the optimum filter exists. However, if the recursive portion of the IIR notch filter is regarded as fixed, the minimization problem becomes a quadratic one.

The main contribution of this paper is to derive a computationally simple estimator based on the minimization of the Least Squares (LS) cost function defined as the output power of a new normalized IIR notch filter. The normalization constant is chosen so as to ensure that the frequency estimate is unbiased. The statistical analysis of the proposed method, of which the RPHD estimator turns out to be a particular case, is derived in terms of the Mean Square Error (MSE) of the frequency estimate.

In essence, the recursive part of the IIR notch filter acts as a data prefilter that enhances the sinusoidal component over the noise background. For this purpose, the prefilter parameter must be tuned to the true frequency value. As this value is *a priori* unknown, an iterative scheme is proposed in which, at each iteration, the prefilter parameter is obtained based on the frequency estimate from the previous step. Simulation results show that convergence is usually achieved in a few iterations, whereas the performance turns out to be very close to the CRLB for a range of Signal to Noise Ratio (SNR) values, provided that the data record length is sufficiently large.

This paper is organized as follows. In Section II, the frequency estimation problem is formulated and the IIR notch filter approach is presented. In Section III, the new normalized IIR notch filter is introduced and the estimator based on the minimization of the corresponding LS cost function is derived. The relationship of the proposed estimator with the PHD method is discussed and an iterative implementation is proposed. Performance is analyzed in Section IV, whereas the selection of the notch filter bandwidth is discussed in Section V. Simulations are presented in Section VI and conclusions are drawn in Section 6.

II. Problem formulation

Consider the problem of estimating the unknown frequency ω_0 of a real-valued sine wave $s(i)$ immersed in white noise $u(i)$. The observed signal, $y(i)$, is given by

$$
\begin{aligned}
y(i) &= s(i) + u(i) \\
&= \alpha \sin(\omega_0 i + \varphi) + u(i), \qquad 1 \leq i \leq N,
\end{aligned}
\tag{1}
$$

where α is the sinusoid amplitude, φ is a random phase uniformly distributed in the interval $[-\pi, \pi[$, $u(i)$ is a real-valued zero mean white noise process with variance σ_u^2, and N is the number of observations. We assume that the processes $u(i)$ and $s(i)$ are statistically independent. The SNR is defined as SNR $\doteq \alpha^2/(2\sigma_u^2)$, and assumed unknown.

Usually, a second-order IIR filter with constrained parameterization is used for notch-filter-based estimation. The following structure is typically used in practice [4]:

$$H_{\tilde{a}_0,r}(z^{-1}) = \frac{A_{\tilde{a}_0}(z^{-1})}{A_{\tilde{a}_0}(rz^{-1})} = \frac{1 + \tilde{a}_0 z^{-1} + z^{-2}}{1 + \tilde{a}_0 r z^{-1} + r^2 z^{-2}}, \tag{2}$$

where $\tilde{a}_0 = -2\cos\tilde{\omega}_0$, $\tilde{\omega}_0$ is the frequency estimate and the parameter r $(0 \leq r < 1)$ is known as the pole contraction factor.

Note that the zeros of $A_{\tilde{a}_0}(z^{-1})$ are on the unit circle, at angular locations $\pm\tilde{\omega}_0$. The basic idea underlying notch-filter-based estimation techniques is the minimization, with respect to $\tilde{a}_0$, of the power of the notch filter output when the input is the observed signal $y(i)$. Due to the presence of $\tilde{a}_0$ in both the numerator and the denominator of (2), this output power is a nonquadratic function of $\tilde{a}_0$. It is known that in the noise-free case this cost function is unimodal, with its only minimum at $\tilde{a}_0 = a_0 \doteq -2\cos\omega_0$ as desired [15], [16]. Seeking this minimum is feasible via e.g. an on-line gradient descent procedure, but obtaining an off-line estimate based on this paradigm is not straightforward. Moreover, the presence of the noise term $u(i)$ in (1) will alter the location of this minimum, unless the pole contraction factor is sufficiently close to one [16].

For these reasons, we consider the minimization of the output power but fixing the denominator in (1), i.e. the notch filter transfer function becomes now

$$H_{\tilde{a}_0,b,r}(z^{-1}) = \frac{A_{\tilde{a}_0}(z^{-1})}{A_b(rz^{-1})} \tag{3}$$

where b is now a fixed parameter. A typical choice for b is the true parameter $a_0 \doteq -2\cos\omega_0$; since a_0 is not available, later on we will present a means to select b.

III. Closed-form estimator based on a normalized IIR notch filter

As a first step in the development of the frequency estimator, the appropriate normalization for the recursive data prefilter is derived. Then the estimate is presented in closed form, together with the iterative procedure for the selection of the prefilter.

A. Choice of normalization factor

Direct minimization of the power of $e(i;\tilde{a}_0) = H_{\tilde{a}_0,b,r}(z^{-1})y(i)$ with respect to $\tilde{a}_0$ leads to a biased estimate of a_0. To see this, write this power as

$$\mathrm{E}\{e^2(i;\tilde{a}_0)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{\tilde{a}_0,b,r}(e^{j\omega})|^2 S_y(\omega)d\omega, \tag{4}$$

where $S_y(\omega)$ is the power spectral density (psd) of $y(i)$, which is the sum of the psd's of $s(i)$ and $u(i)$, namely

$$S_y(\omega) = S_s(\omega) + S_u(\omega) \tag{5}$$

where

$$S_s(\omega) = \frac{\alpha^2}{4}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right], \qquad S_u(\omega) = \sigma_u^2. \tag{6}$$

Therefore the output power (4) can be split into the signal and noise induced terms:

$$\begin{aligned} \mathrm{E}\{e^2(i;\tilde{a}_0)\} &= J_s(\tilde{a}_0) + J_u(\tilde{a}_0), &(7)\\ J_s(\tilde{a}_0) &= \frac{\alpha^2}{4\pi}|H_{\tilde{a}_0,b,r}(e^{j\omega_0})|^2, &(8)\\ J_u(\tilde{a}_0) &= \frac{\sigma_u^2}{2\pi}\int_{-\pi}^{\pi}|H_{\tilde{a}_0,b,r}(e^{j\omega})|^2 d\omega. &(9) \end{aligned}$$

Substituting the expression of the notch filter $H_{\tilde{a}_0,b,r}(z^{-1})$ one obtains

$$\begin{aligned} J_s(\tilde{a}_0) &= \frac{\alpha^2}{4\pi}\frac{(\tilde{a}_0 - a_0)^2}{(1-r^2)^2 - r(1+r^2)a_0 b + r^2(a_0^2 + b^2)}, &(10)\\ J_u(\tilde{a}_0) &= \sigma_u^2 \frac{(1+r^2)\tilde{a}_0^2 - 4r\tilde{a}_0 b + 2r^2b^2 - 2r^4 + 2}{(1-r^2)\left[(1+r^2)^2 - r^2b^2\right]}. &(11) \end{aligned}$$

Note that the signal induced term $J_s(\tilde{a}_0)$ is minimized when $\tilde{a}_0 = a_0$, as desired. On the other hand, since the noise induced term $J_u(\tilde{a}_0)$ is a function of $\tilde{a}_0$, the minimum of $\mathrm{E}\{e^2(i;\tilde{a}_0)\}$ will depend on the noise power σ_u^2. As a result, minimizing (4) in the presence of noise will result in a biased frequency estimate. This bias problem would be avoided if the noise induced term could be made independent of the parameter $\tilde{a}_0$. To do so, and in view of (11), we propose to normalize the IIR notch filter by the following strictly positive function of $\tilde{a}_0$, parameterized by b and r (with $r \in [0, 1[$ and $b \in [-2, 2]$):

$$M_{\tilde{a}_0,b,r} \doteq \sqrt{(1+r^2)\tilde{a}_0^2 - 4r\tilde{a}_0 b + 2r^2b^2 - 2r^4 + 2}. \tag{12}$$

This results in a new IIR notch filter

$$\bar{H}_{\tilde{a}_0,b,r}(z^{-1}) = \frac{H_{\tilde{a}_0,b,r}(z^{-1})}{M_{\tilde{a}_0,b,r}}. \tag{13}$$

The expression of the power of the corresponding output $\bar{e}(i;\tilde{a}_0) = \bar{H}_{\tilde{a}_0,b,r}(z^{-1})y(i)$ is now

$$\mathrm{E}\{\bar{e}^2(i;\tilde{a}_0)\} = \bar{J}_s(\tilde{a}_0) + \bar{J}_u, \tag{14}$$

$$\bar{J}_s(\tilde{a}_0) = \frac{J_s(\tilde{a}_0)}{M^2_{\tilde{a}_0,b,r}}, \qquad \bar{J}_u = \frac{J_u(\tilde{a}_0)}{M^2_{\tilde{a}_0,b,r}} \tag{15}$$

By design, $\bar{J}_u$ is independent of $\tilde{a}_0$. On the other hand, the global minimum of $\bar{J}_s(\tilde{a}_0)$ is still located at $\tilde{a}_0 = a_0$. Although $\bar{J}_s(\tilde{a}_0)$ is no longer a quadratic function of $\tilde{a}_0$ (in contrast to $J_s(\tilde{a}_0)$), it turns out that it still enjoys a unique minimum at $\tilde{a}_0 = a_0$. This can be seen by noting that the only solution different from $\tilde{a}_0 = a_0$ of $\partial\bar{J}_s(\tilde{a}_0)/\partial\tilde{a}_0 = 0$ is

$$\tilde{a}_0 = \frac{2(1 - rb + r^2b^2 - r^4)}{4rb - (1+r^2)a_0}, \tag{16}$$

which can be easily shown to correspond to a *maximum* of $\bar{J}_s(\tilde{a}_0)$.

B. Frequency estimator

In order to obtain a frequency estimator in closed form based on the ideas of the previous subsection, we adopt an LS approach.

Given b, r, we define as before the output of the normalized IIR notch filter as $\bar{e}(i;\tilde{a}_0) = \bar{H}_{\tilde{a}_0,b,r}(z^{-1})y(i)$. The empirical power is then obtained as

$$P_N(\tilde{a}_0) \doteq \frac{1}{N-2}\sum_{i=3}^{N}\bar{e}^2(i;\tilde{a}_0). \tag{17}$$

In order to obtain the minimizer of (17) we must solve

$$\sum_{i=3}^{N} \bar{e}(i;\tilde{a}_0)\frac{\partial \bar{e}(i;\tilde{a}_0)}{\partial \tilde{a}_0} = 0. \tag{18}$$

To do so, let $w(i)$ be the signal obtained by prefiltering the data $y(i)$ with the all-pole filter $1/A_b(rz^{-1})$. Then we can write

$$\bar{e}(i;\tilde{a}_0) = \frac{1}{M_{\tilde{a}_0,b,r}}\left[w(i) + \tilde{a}_0 w(i-1) + w(i-2)\right], \tag{19}$$

from which one readily obtains

$$\begin{aligned} \frac{\partial \bar{e}(i;\tilde{a}_0)}{\partial \tilde{a}_0} &= \frac{1}{M^3_{\tilde{a}_0,b,r}}\Big[(2rb-(1+r^2)\tilde{a}_0)(w(i)+w(i-2)) \\ &\quad + 2(1-rb\tilde{a}_0+r^2b^2-r^4)w(i-1)\Big]. \end{aligned} \tag{20}$$

Thus the LS minimizer must satisfy the quadratic

$$\vartheta_N \tilde{a}_0^2 + \eta_N \tilde{a}_0 - 2\varrho_N = 0, \tag{21}$$

where

$$\eta_N \doteq \sum_{i=3}^{N}\Big[(1+r^2)\left(w(i)+w(i-2)\right)^2 - 2(1+r^2b^2-r^4+2)x^2(i-1)\Big], \tag{22}$$

$$\vartheta_N \doteq \sum_{i=3}^{N}\left[2rbx^2(i-1) + (1+r^2)w(i-1)\left(w(i)+w(i-2)\right)\right], \tag{23}$$

$$\varrho_N \doteq \sum_{i=3}^{N}\Big[rb\left(w(i)+w(i-2)\right)^2 + (1+r^2b^2-r^4)w(i-1)\left(w(i)+w(i-2)\right)\Big]. \tag{24}$$

Of the two roots of (21), one corresponds to a maximum of $P_N(\tilde{a}_0)$ and the other one is the solution sought:

$$\tilde{a}_0 = -\frac{\eta_N + \sqrt{\eta_N^2 + 8\vartheta_N\varrho_N}}{2\vartheta_N}, \tag{25}$$

which provides the desired frequency estimate as

$$\tilde{\omega}_0 = \arccos\left(-\frac{\hat{a}_0}{2}\right), \qquad \text{with} \quad \hat{a}_0 \doteq \operatorname{sgn}(\tilde{a}_0)\cdot\min\{2,|\tilde{a}_0|\} \tag{26}$$

For sufficiently large N, a simplified version of (25) can be derived. Note that for large N, the following hold:

$$\frac{1}{N-2}\sum_{i=3}^{N} x^2(n) \approx \frac{1}{N-2}\sum_{i=3}^{N} x^2(n-1) \approx \frac{1}{N-2}\sum_{i=3}^{N} x^2(n-2), \tag{27}$$

$$\frac{1}{N-2}\sum_{i=3}^{N} w(n)w(n-1) \approx \frac{1}{N-2}\sum_{i=3}^{N} w(n-1)w(n-2). \tag{28}$$

Hence, the estimate (25) asymptotically converges to

$$\tilde{a}_0 = -\frac{r^2\left(1+r^2-b^2\right)\rho_0 + \left(1+r^2\right)\rho_2 + \sqrt{\Delta}}{2\left[rb\rho_0 + \left(1+r^2\right)\rho_1\right]}, \tag{29}$$

where

$$\begin{aligned} \Delta &\doteq \left[r^2\left(1+r^2-b^2\right)\rho_0 + \left(1+r^2\right)\rho_2\right]^2 \\ &\quad + 8\left[rb\rho_0 + \left(1+r^2\right)\rho_1\right]\left[rb\left(\rho_0+\rho_2\right) + \left(1+r^2b^2-r^4\right)\rho_1\right], \end{aligned} \tag{30}$$

and with the empirical autocorrelation coefficients defined as

$$\rho_p \doteq \frac{1}{N-2}\sum_{i=3}^{N} w(i)w(i-p) \quad \text{for} \quad p = 0,\, 1,\, 2. \tag{31}$$

In order to check the unbiasedness of the proposed estimator, note that the prefiltered signal $w(i)$ consists of a sinusoid in colored noise, and that the empirical autocorrelation estimates (31) asymptotically converge to the true values

$$\lim_{N\to\infty} \rho_p = \frac{\cos(p\omega_0)}{|A_b\left(re^{j\omega_0}\right)|^2} + \frac{\sigma_u^2}{2\pi j}\oint \frac{z^{p-1}dz}{A_b\left(rz^{-1}\right)A_b\left(rz\right)}. \tag{32}$$

Using the residue theorem for the evaluation of the integral in (32), it is found that

$$\lim_{N\to\infty} \rho_0 = \left(1+r^2\right)\xi_{r,b} + \varrho_{a_0,b,r}, \tag{33}$$

$$\lim_{N\to\infty} \rho_1 = -rb\xi_{r,b} - \frac{a_0}{2}\varrho_{a_0,b,r}, \tag{34}$$

$$\lim_{N\to\infty} \rho_2 = -r^2\left(r^2-b^2+1\right)\xi_{r,b} + \left(\frac{a_0^2}{2}-1\right)\varrho_{a_0,b,r}, \tag{35}$$

with

$$\xi_{r,b} \doteq \frac{\sigma_u^2}{\left(1-r^2\right)\left(\left(1+r^2\right)^2 - r^2b^2\right)}, \tag{36}$$

$$\varrho_{a_0,b,r} \doteq \frac{\alpha^2}{2\left(\left(1-r^2\right)^2 - r\left(1+r^2\right)a_0b + r^2\left(a_0^2+b^2\right)\right)}. \tag{37}$$

Substituting in (29) the asymptotic values of ρ_p given by (33)-(35), it is found that $\tilde{a}_0$ converges to $a_0 = -2\cos\left(\omega_0\right)$. Thus the proposed estimator is asymptotically unbiased.

C. *Relation with the PHD estimator*

In the case of a single real-valued tone, the PHD estimate [9], [10], [12] is obtained in terms of the unit-norm eigenvector $[v_0\ v_1\ v_2]^T$ corresponding to the smallest eigenvalue of the 3×3 sample autocorrelation matrix

$$\begin{pmatrix} \check{\rho}_0 & \check{\rho}_1 & \check{\rho}_2 \\ \check{\rho}_1 & \check{\rho}_0 & \check{\rho}_1 \\ \check{\rho}_2 & \check{\rho}_1 & \check{\rho}_0 \end{pmatrix},$$

where $\check{\rho}_p$ denotes the lag-p sample autocorrelation coefficient of the observations $y(i)$, given by:

$$\check{\rho}_p = \frac{1}{N-p}\sum_{i=p+1}^{N} y(i)y(i-p). \tag{38}$$

The corresponding eigenvector is symmetric ($v_2 = v_0$), and the frequency estimate is taken as the angular position $\hat{w}$ of the zero of the transfer function $v_0 + v_1z^{-1} + v_0z^{-2}$ [12]

$$\hat{w} = \arccos\left(\frac{\check{\rho}_2 + \sqrt{\check{\rho}_2^2 + 8\check{\rho}_1^2}}{4\check{\rho}_1}\right). \tag{39}$$

In the same vein, a similar estimator was derived in [13] based on the linear prediction property of sinusoidal signals. Termed *Reformed* PHD (RPHD), it is obtained under a constrained LS criterion and is given by

$$\hat{w} = \arccos\left(\frac{\gamma_N + \sqrt{\gamma_N^2 + 8\beta_N^2}}{4\beta_N}\right), \tag{40}$$

where

$$\gamma_N \doteq \sum_{i=3}^{N}\Big([y(i)+y(i-2)]^2-2y^2(i-1)\Big), \tag{41}$$

$$\beta_N \doteq \sum_{i=3}^{N}[y(i)+y(i-2)]\,y(i-1). \tag{42}$$

Consider now the IIR notch filter-based closed-form estimator (25)-(26). It turns out that for the particular case $r=0$ (i.e. when no data prefiltering is applied), the frequency estimate (26) corresponds to the RPHD estimate (40), whereas the estimate obtained by using the asymptotic expression (29) in place of (25) corresponds to the PHD estimate (39). Hence, both PHD and RPHD are obtained as particular cases of the proposed scheme.

D. *Iterative update of the data prefilter*

Observe that, if properly chosen, the data prefilter $1/A_b(rz^{-1})$ has the potential to enhance the desired frequency over the noise, thus improving the effective SNR. So far we have assumed that the parameter b of the prefilter is fixed *a priori*. Note that this parameter determines the angular position of the poles of the data prefilter, and thus ideally one would like to choose $b=a_0=-2\cos\omega_0$. Since ω_0 is of course not available, it seems natural to use an iterative scheme in which the prefilter is designed at each iteration based on the frequency estimate obtained in the previous step. In the first iteration, $r=0$ can be used (and thus the RPHD estimate is used to initialize the recursion), after which a suitable value $r>0$ is set. The iterative procedure is summarized as follows:

1) Obtain an initial estimate $\tilde{a}_0^{(0)}$ based on RPHD (i.e. (25) with $r=0$).

Set r to a suitable value $r\in(0,1)$. For $k=1,\,2,\,3,\,\dots$

2) Set $b^{(k)}=\tilde{a}_0^{(k-1)}$ and obtain the prefiltered data $w(i)=[1/A_{b^{(k)}}(rz^{-1})]y(i)$.
3) Compute the new estimate $\tilde{a}_0^{(k)}$ using (25).
4) Repeat Steps 2 and 3 until convergence. Obtain the frequency estimate $\tilde{\omega}_0$ via (26).

IV. Performance Analysis

By a Taylor series argument, and assuming small errors, the MSEs of the estimates $\tilde{\omega}_0$ and $\tilde{a}_0=-2\cos\tilde{\omega}_0$ are easily shown to be related by

$$\text{MSE}(\tilde{\omega}_0)\simeq\frac{\text{MSE}(\tilde{a}_0)}{4\sin^2\omega_0},\qquad \omega_0\notin\{0,\pi\}. \tag{43}$$

First we derive the MSE of $\tilde{a}_0$ for a fixed value of b. Once this is done, a recursion for the MSE of $\tilde{a}_0^{(k)}$ at iteration k will be obtained, which will yield the desired MSE value after convergence.

A. *MSE for fixed b*

For given values of b and r, the estimate $\tilde{a}_0$ must satisfy

$$F_N(\tilde{a}_0)\doteq\frac{1}{N-2}\sum_{i=3}^{N}\bar{e}\,(i;\tilde{a}_0)\,\psi\,(i;\tilde{a}_0)=0, \tag{44}$$

where $\psi\,(i;\tilde{a}_0)=[\partial\bar{e}\,(i;a)/\partial a]_{a=\tilde{a}_0}$. Assuming small errors, we can use a first-order expansion of $F_N(\tilde{a}_0)$ around a_0

$$F_N(\tilde{a}_0)\simeq F_N(a_0)+\beta\,(\tilde{a}_0-a_0)\,, \tag{45}$$

where

$$\beta \doteq \left. \frac{\partial F_N(a)}{\partial a} \right|_{a=a_0} . \tag{46}$$

Using the weak law of large numbers, the evaluation of β for sufficiently large N yields

$$\beta = \frac{\alpha^2}{2M_{a_0,b,r}^2 \left[(1-r^2)^2 - r\,(1+r^2)\,a_0 b + r^2\,(a_0^2+b^2) \right]} . \tag{47}$$

The terms neglected in (45) go to zero faster than $|\tilde{a}_0 - a_0|$ when N is sufficiently large.

It follows from (45) that the MSE is

$$\text{MSE}\,(\tilde{a}_0|\,b) = \text{E}\left[(\tilde{a}_0 - a_0)^2 \,|\, b \right] \simeq \frac{\text{E}[F_N^2(a_0)]}{\beta^2} . \tag{48}$$

The evaluation of $\text{E}\left[F_N^2(a_0)\right]$ for large N is done in the Appendix. The resulting asymptotic expression for the MSE is

$$\text{MSE}\,(\tilde{a}_0|\,b) \simeq \frac{2}{N \cdot \text{SNR}^2} \cdot \frac{C_{a_0,b,r} \left[\left(1-r^2\right)^2 - r\left(1+r^2\right) a_0 b + r^2 \left(a_0^2+b^2\right) \right]^2}{M_{a_0,b,r}^4\,(1-r^2)\,[b^2 r^2 - (1+r^2)^2]} , \tag{49}$$

where

$$\begin{aligned} C_{a_0,b,r} \quad \doteq \quad & 2\left[a_0 br(1+r^2) - (1-r^2)^2 - (a_0^2+b^2)r^2\right]^2 \times \left\{(4-3a_0^2+a_0^4) + 4b(a_0-a_0^3)r \right. \\ & - [4(1+b^2) - (3+7b^2)a_0^2 - a_0^4]r^2 - 4b[(2+b^2)a_0 + a_0^3]r^3 \\ & \left. -[4(1-3b^2) - (7+3b^2)a_0^2]r^4 - 12ba_0 r^5 + (4+a_0^2)r^6 \right\} . \end{aligned} \tag{50}$$

For the particular case $r = 0$, (49) reduces to the following known approximation for the MSE of the RPHD estimator [13]:

$$\text{MSE}_{\text{RPHD}} \simeq \frac{4\left[a_0^4 - 3a_0^2 + 4\right]}{N\text{SNR}^2(a_0^2+2)^2} . \tag{51}$$

B. Evolution of the MSE

The previous analysis assumed b was fixed. At step k of the proposed iterative estimation scheme, the MSE of $\tilde{a}_0^{(k)}$ will be approximately given by the expected value of (49) after setting $b = b^{(k)}$:

$$\text{MSE}\left(\tilde{a}_0^{(k)}\right) \simeq \text{E}\left\{\text{MSE}\left(\tilde{a}_0^{(k)} \big|\, b^{(k)}\right)\right\} \tag{52}$$

Since the iteration sets $b^{(k)} = \tilde{a}_0^{(k-1)}$, we assume that $b^{(k)} = a_0 + \varepsilon^{(k)}$, with $\varepsilon^{(k)}$ a zero mean random variable with variance

$$\text{E}\left\{(\varepsilon^{(k)})^2\right\} = \text{E}\left\{(b^{(k)} - a_0)^2\right\} = \text{MSE}\left(\tilde{a}_0^{(k-1)}\right) . \tag{53}$$

Let $f(b) \doteq \text{MSE}\left(\tilde{a}_0^{(k)} \big|\, b\right)$, and consider a second-order expansion of $f(b^{(k)})$ around a_0:

$$f(b^{(k)}) \simeq f(a_0) + \varepsilon^{(k)} \left. \frac{\partial f(b)}{\partial b} \right|_{b=a_0} + \frac{(\varepsilon^{(k)})^2}{2} \left. \frac{\partial^2 f(b)}{\partial b^2} \right|_{b=a_0} . \tag{54}$$

Taking the expectation of (54), one has

$$\text{MSE}\left(\tilde{a}_0^{(k)}\right) \simeq f(a_0) + \frac{1}{2}\text{MSE}\left(\tilde{a}_0^{(k-1)}\right) \left. \frac{\partial^2 f(b)}{\partial b^2} \right|_{b=a_0} . \tag{55}$$

At the first iteration, $\tilde{a}_0^{(0)}$ coincides with the RPHD estimate, and hence $\text{MSE}\left(\tilde{a}_0^{(0)}\right)$ is given by (51).

The evaluation of $f(a_0)$ shows that

$$f(a_0) = \frac{(1-r)^3 h_0(a_0, r)}{N\,\mathrm{SNR}^2}, \tag{56}$$

where

$$h_0(a_0, r) = \frac{4[(1+r)^2 - a_0^2 r]^2 \left[4(1+r^2)(1+r)^2 + a_0^2(r^4 - 10r^3 - 2r^2 - 2r - 3) + a_0^4(3r^2 - 2r + 1)\right]}{(1+r)[(1+r^2)^2 - a_0^2 r^2]\left[2(1+r)(1+r^2) - 3a_0^2 r + a_0^2\right]^2}.$$

On the other hand,

$$\left.\frac{\partial^2 f(b)}{\partial b^2}\right|_{b=a_0} = \frac{r^2(1-r)h_1(a_0, r)}{N\,\mathrm{SNR}^2}, \tag{57}$$

where

$$h_1(a_0, r) = \frac{\sum_{m=0}^{7} p_m(r) a_0^{2m}}{(1+r)(2r^3 + 2r^2 - 3a_0^2 r + 2r + a_0^2 + 2)^4 (1 + 2r^2 + r^4 - r^2 a_0^2)^3}. \tag{58}$$

The terms $p_m(r)$ are polynomial functions of r of degree up to 20.[1]

As k goes to ∞, the MSE at convergence can be obtained from (55) since

$$\mathrm{MSE}\left(\tilde{a}_0^{(\infty)}\right) \simeq f(a_0) + \frac{1}{2}\mathrm{MSE}\left(\tilde{a}_0^{(\infty)}\right) \left.\frac{\partial^2 f(b)}{\partial b^2}\right|_{b=a_0}. \tag{59}$$

Solving equation (59) and using (43), one finally has

$$\mathrm{MSE}\left(\tilde{\omega}_0^{(\infty)}\right) \simeq \frac{2(1-r)^3 h_0(a_0, r)}{[2N\,\mathrm{SNR}^2 - r^2(1-r)h_1(a_0, r)](4 - a_0^2)}. \tag{60}$$

C. Numerical examples

Computer simulations had been carried out to validate the theoretical MSE of the proposed estimator. The magnitude of the input sine wave is set to $\sqrt{2}$ and its phase is randomly selected in $[0, 2\pi[$. All results are averages of 10^3 independent experiments.

The theoretical MSE at convergence (60) as a function of r and for different sequence lengths N is shown in Figs. 1 and 2, for SNR $= 10$ and 0 dB respectively. The sinusoid frequency is $\omega_0 = 0.4\pi$. Also shown are the empirical results obtained after three iterations of the proposed estimator. This empirical MSE agrees reasonably well with the theoretical expression, more so for large N, as expected. We also note that, as r approaches 1, this match requires larger values of N. Observe that considerable improvements with respect to RPHD ($r = 0$) can be achieved by selecting $r > 0$.

Fig. 3 shows the theoretical and empirical MSE versus N in a very noisy setting (SNR=-10 dB), again for $\omega_0 = 0.4\pi$ and with $r = 0.8$. It is observed that convergence is achieved in two to three iterations. Even in this low SNR setting a good match between the theoretical and experimental values is obtained if N is large enough.

[1]The exact expressions of $p_m(r)$ are available upon request.

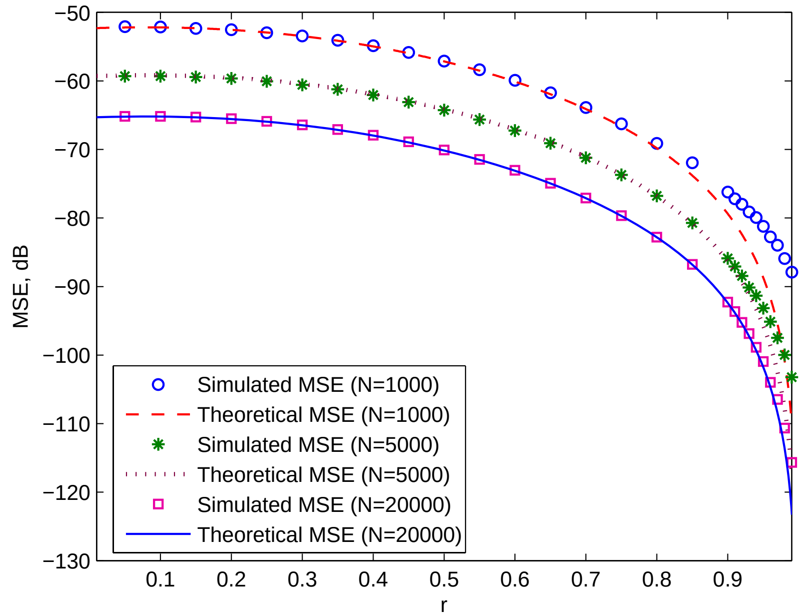

Fig. 1. Mean squared frequency errors versus r, SNR=10 dB, $\omega_0 = 0.4\pi$.

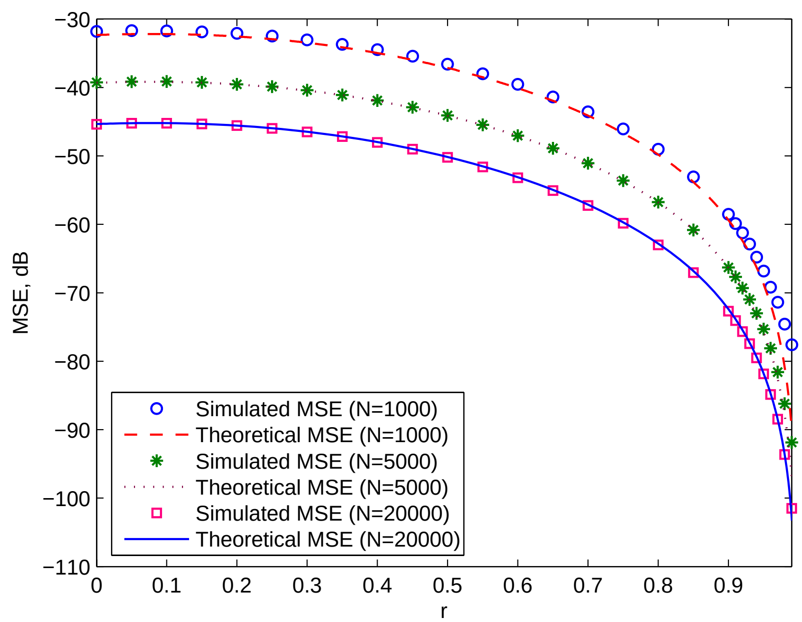

Fig. 2. Mean squared frequency errors versus r, SNR=0 dB, $\omega_0 = 0.4\pi$.

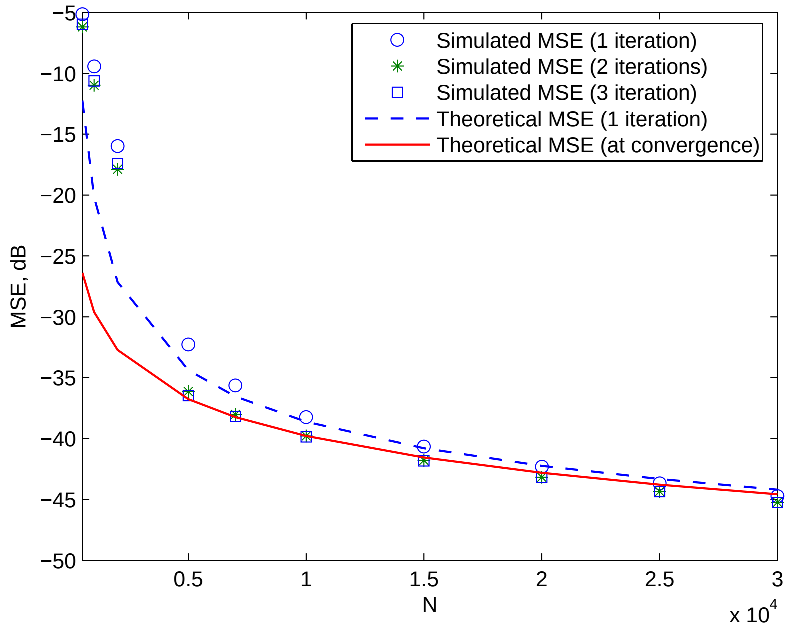

Fig. 3. Mean squared frequency errors versus N, SNR=-10 dB, $r = 0.8$, $\omega_0 = 0.4\pi$.

V. Selection of the pole contraction factor

As seen above, a proper choice of r can improve noticeably the performance of the estimate. Fig. 4 shows the behavior of the theoretical MSE in terms of r for $\omega_0 = 0.4\pi$, $N = 5000$ and different SNR levels. The optimal parameter r_{opt} minimizing the MSE depends on the SNR. The higher the SNR, the larger r_{opt} is for a given iteration number. On the other hand, if the SNR is sufficiently low, a value of r too close to 1 in the first iteration can result in degraded performance. Thus in low SNR settings the performance of the proposed estimator becomes sensitive to the initial choice of the parameter r. Fig. 5 plots the theoretical MSE versus r for $\omega_0 = 0.4\pi$, N=10000, SNR $= 0$ dB. It is seen that r_{opt} also depends on the sequence length N.

Fig. 6 and Fig. 7 display r_{opt} as a function of the iteration number for different N and SNR levels, and for $\omega_0 = 0.4\pi$ and 0.1π respectively. It is seen that r_{opt} increases with the iteration number. The initial and final values of r_{opt} also depend on ω_0, N, and the SNR level.

These observations motivate the selection of a different pole contraction factor $r^{(k)}$ at different iterations. In practice, no *a priori* information is available on the input sine wave so that its frequecy may fall outside of the prefilter passband, especially for small SNR and/or short data lengths. Therefore it makes sense to use a wider passband at the first iteration that becomes narrower as iterations evolve. This 'bandwidth thinning' strategy is common in the design of adaptive notch filters [15]. A simple way to do this is to let r grow exponentially from $r^{(1)}$ to a final value $r^{(\infty)}$ according to

$$r^{(k+1)} = \lambda r^{(k)} + (1-\lambda) r^{(\infty)}, \qquad 0 < \lambda < 1. \tag{61}$$

The parameter λ determines the change rate of $r^{(k)}$, which should be larger for large data lengths in order to speed up convergence. On the other hand, with short data records a slower variation of r should help increase sensitivity to the presence of the sine wave. Thus, λ should somehow be inversely proportional to N.

In our simulation studies, we apply the algorithm with the numerical values

$r^{(1)} = 0.75; \quad r^{(\infty)} = 0.995.$

The value $r^{(1)} = 0.75$ was chosen as the smallest pole contraction which has significant effect on the bandwidth of the notch filter, corresponding to the worst situation, where small sequence length N is available and low SNR environment is considered. The value $r^{(\infty)} = 0.995$ was found to yield most accurate results in estimating the sine wave frequency.

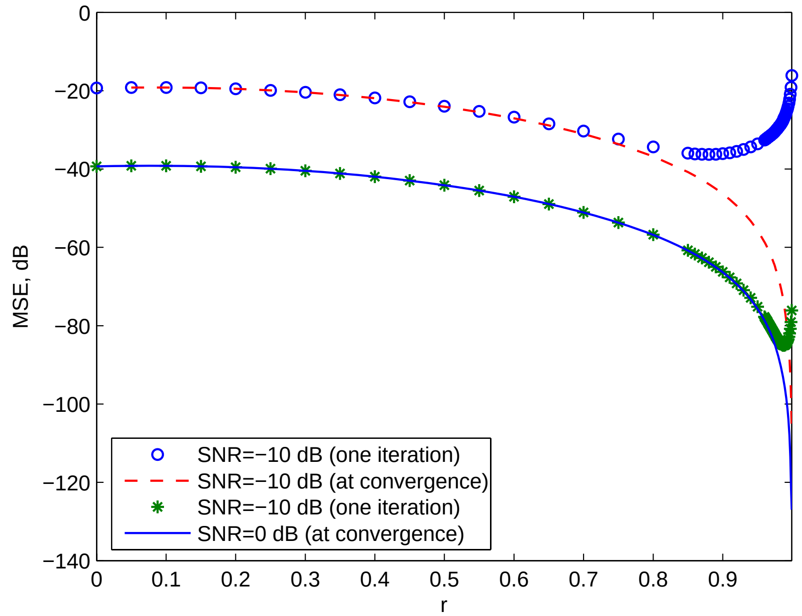

Fig. 4. Mean squared frequency errors versus r, $\omega_0 = 0.4\pi$, N=5000.

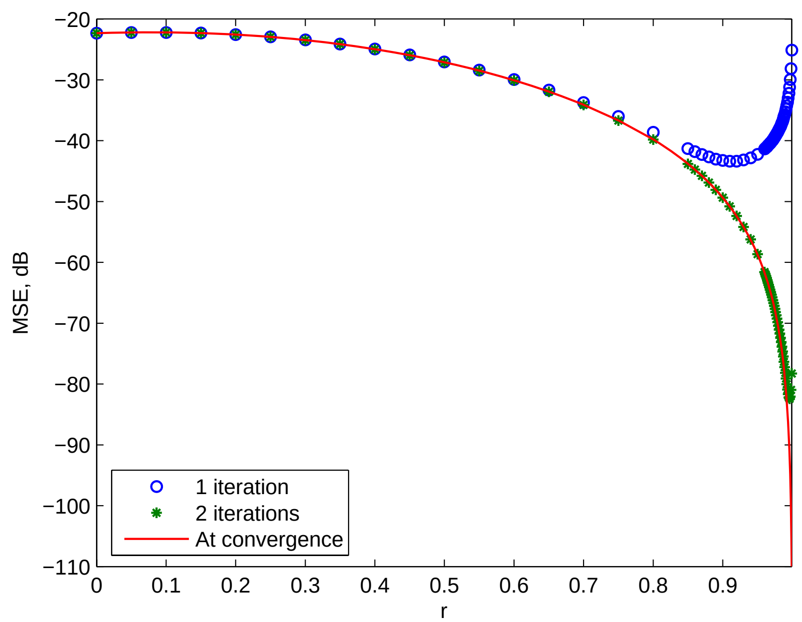

Fig. 5. Mean squared frequency errors versus r, $\omega_0 = 0.4\pi$, SNR=-10 dB, N=10000.

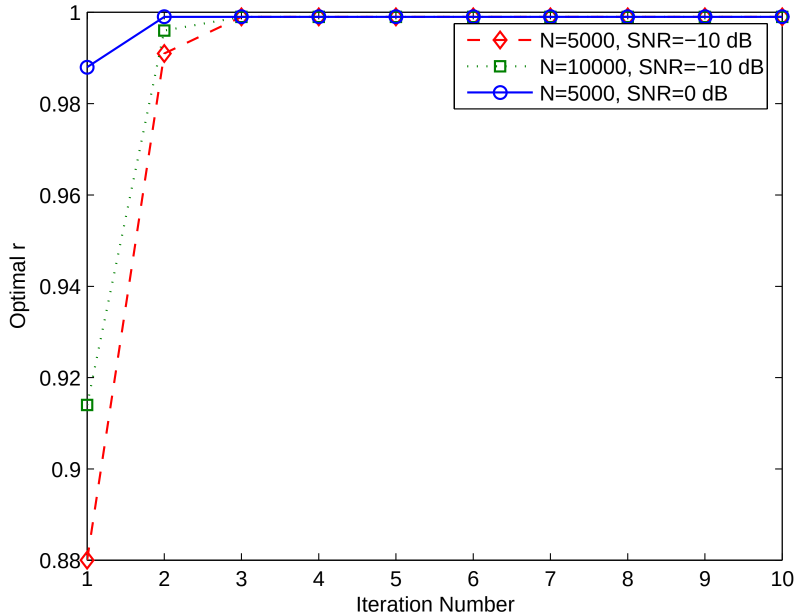

Fig. 6. Optimal values of r versus iteration number, $\omega_0 = 0.4\pi$.

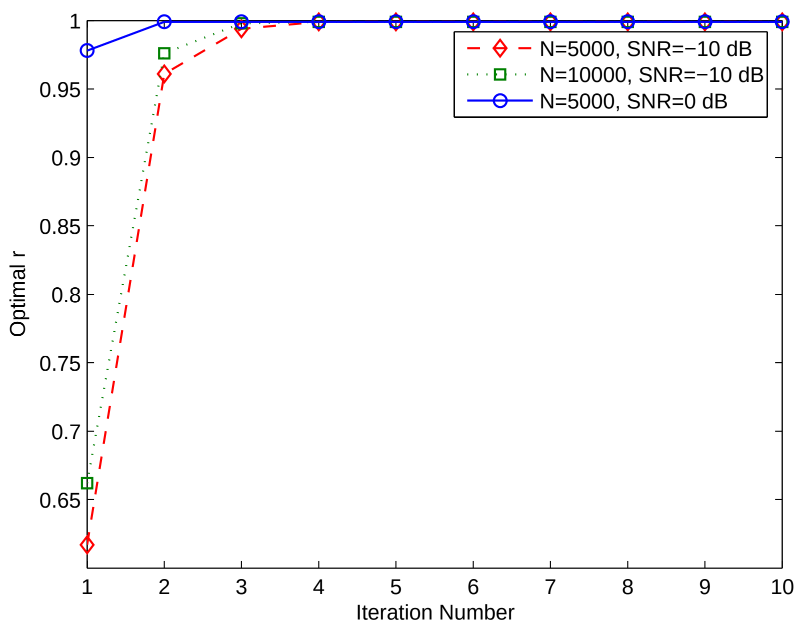

Fig. 7. Optimal values of r versus iteration number, $\omega_0 = 0.1\pi$.

VI. Simulation results

In the following experiments we have made use of the following parameter values for the adjustment of the pole contraction factor: $r^{(1)} = 0.75$, $r^{\infty} = 0.995$, and

$$\lambda(N) = 0.93/(1 + (N/200)^2). \tag{62}$$

Fig. 8 shows the MSE along the iterations for $\omega_0 = 0.4\pi$, setting SNR $= 10$ dB, and for $N = 20$, 200 and 1000. Similarly, in Fig. 9 we fix $N = 200$ and consider SNR $= 0$, 10 and 20 dB. Also shown is the (approximate) CRLB for this frequency estimation problem [17]:

$$\mathrm{var}_{\mathrm{CR}}\left(\tilde{\omega}_0\right) \cong \frac{12}{N^3\,\mathrm{SNR}}. \tag{63}$$

A noticeable improvement can be observed with just one iteration; furthermore, convergence is achieved in about four iterations, and the achieved MSE is very close to the CRLB.

Next we compare the proposed estimator (using four iterations) with the RPHD [13] and ML [3] estimators. Fig. 10 shows the MSE versus SNR for $\omega_0 = 0.4\pi$ and $N = 200$. The ML estimate performs very close to the theoretical optimum in the whole range SNR$\in [-5, 20]$ dB considered. The proposed estimate is also very close to this optimum, except at very low SNR, and with a much smaller computational cost. The MSE of the RPHD estimate is considerably far above the CRLB.

Fig. 11 plots the MSE versus N for $\omega_0 = 0.4\pi$, SNR=10 dB. Again, in addition to outperforming RPHD, the MSE achieved by the proposed estimate is very close to the CRLB as soon as $N > 20$.

Fig. 12 shows the variation of the MSE as a function of ω_0 for SNR $= 10$ dB and $N = 20$. The proposed estimate performs close to the CRLB for ω_0 close to 0.5π, whereas the ML method attains this bound for $\omega_0 \in [0.1\pi, 0.9\pi]$. Fig. 13 shows the corresponding results when the data length is increased to $N = 200$. The range of frequencies over which the proposed estimate attains the CRLB has significantly expanded. These results highlight the good behavior of the proposed method in high SNR even for short data records.

Fig. 14 shows the MSE as function of ω_0 for $N = 1000$ in a noisy setting for which SNR $= 0$ dB. Performance is comparable to that of the ML estimator, and to the CRLB (except near the critical frequencies $\{0, \pi\}$). This shows that the proposed estimator can yield optimum estimation performance even with low SNR levels provided the data length is sufficiently large.

From Figs. 10-14, we can conclude that the proposed estimator is superior to the RPHD estimator and can achieve very good estimation performance. It can approach the CRLB for sufficiently high SNR and/or high data lengths.

VII. Conclusions

An asymptotically unbiased estimator for single sinusoid detection in the presence of white noise was presented. This new estimator minimizes the LS cost function given by the output of a normalized IIR notch filter. It was shown that the estimator variance can approach the CRLB under different SNR levels if the number of data points is sufficiently large. The accuracy of this estimator and its computational simplicity make it attractive in practice and is a motivation for further generalization to the case of multiple sinusoid detection.

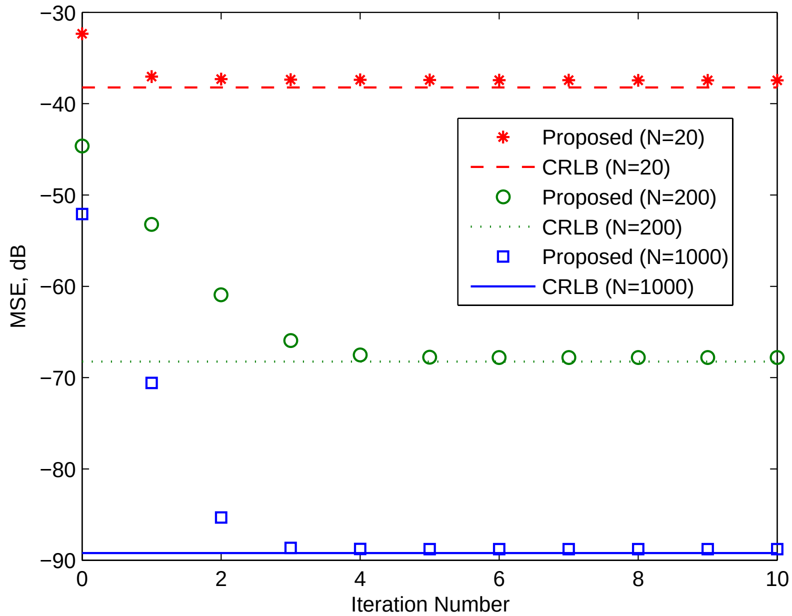

Fig. 8. Mean squared frequency errors versus iteration number for $\omega_0 = 0.4\pi$, SNR=10 dB.

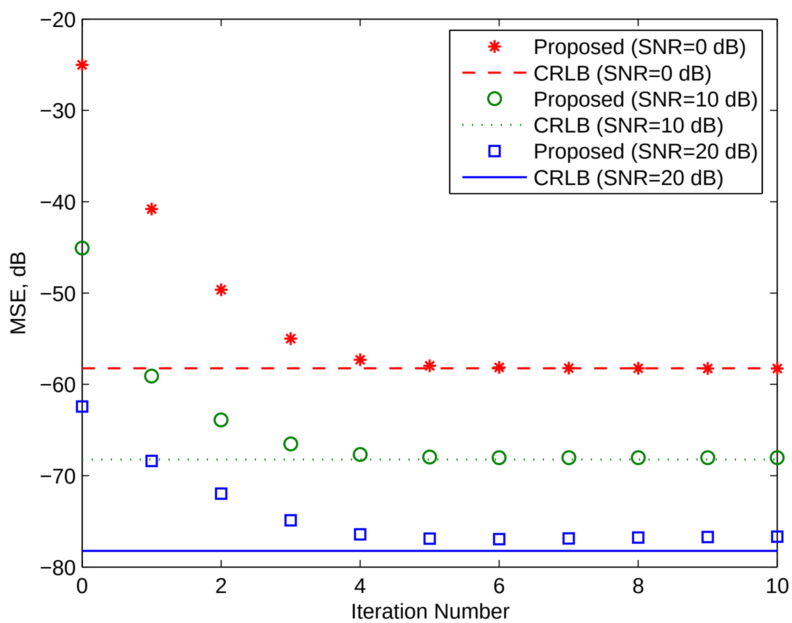

Fig. 9. Mean squared frequency errors versus iteration number for $\omega_0 = 0.4\pi$, N=200.

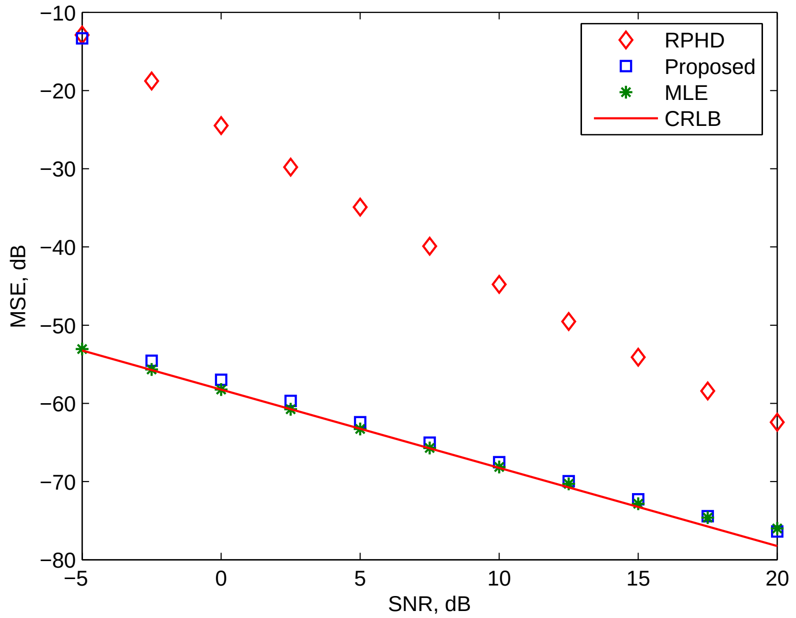

Fig. 10. Mean squared frequency errors versus SNR, $\omega_0 = 0.4\pi$, N=200.

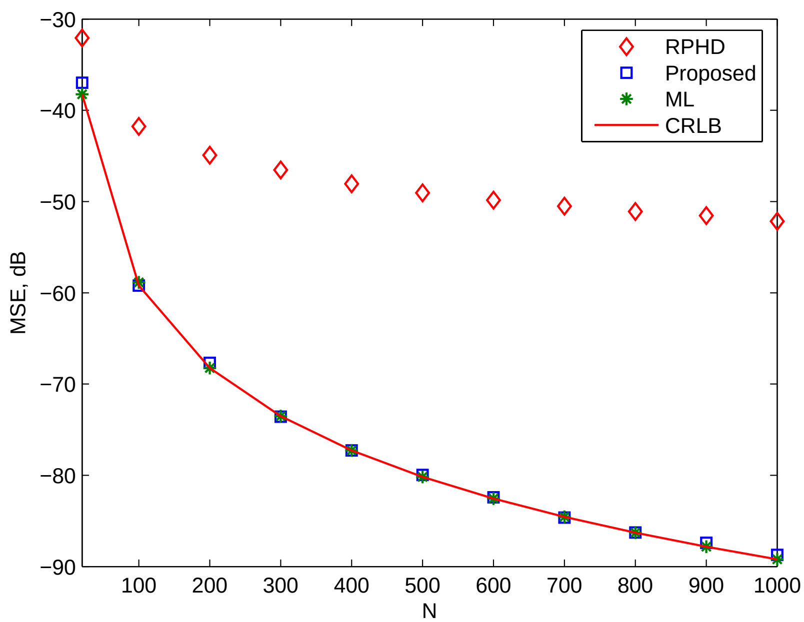

Fig. 11. Mean squared frequency errors versus N, $\omega_0 = 0.4\pi$, SNR = 10 dB.

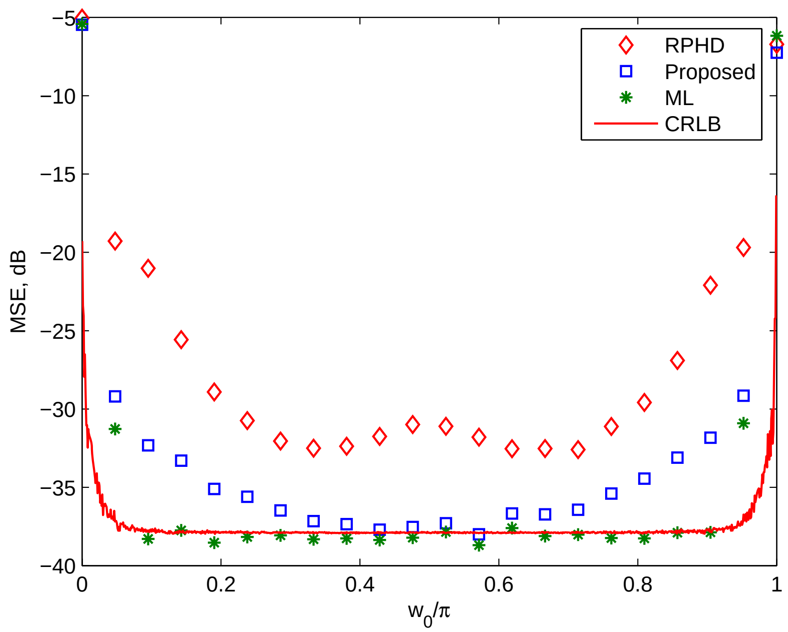

Fig. 12. Mean squared frequency errors versus ω_0, SNR=10 dB, N=20.

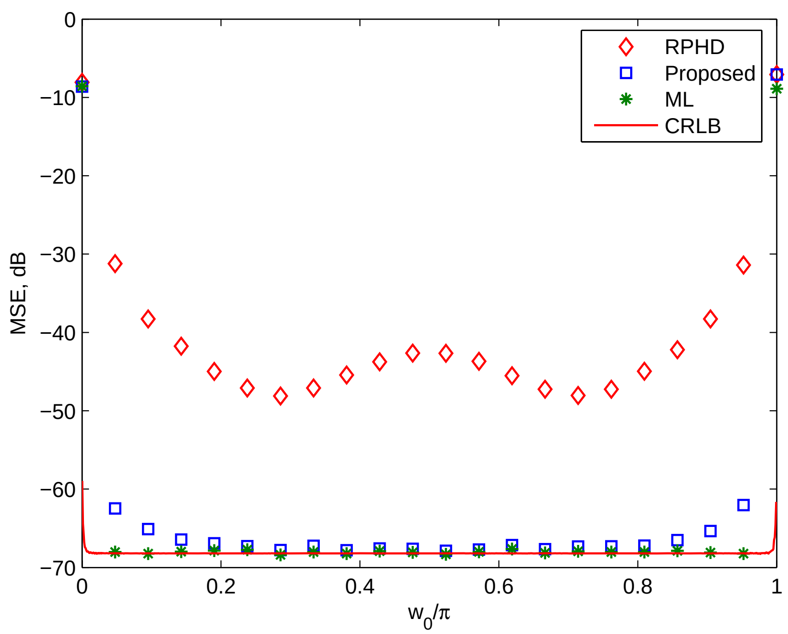

Fig. 13. Mean squared frequency errors versus ω_0, SNR=10 dB, N=200.

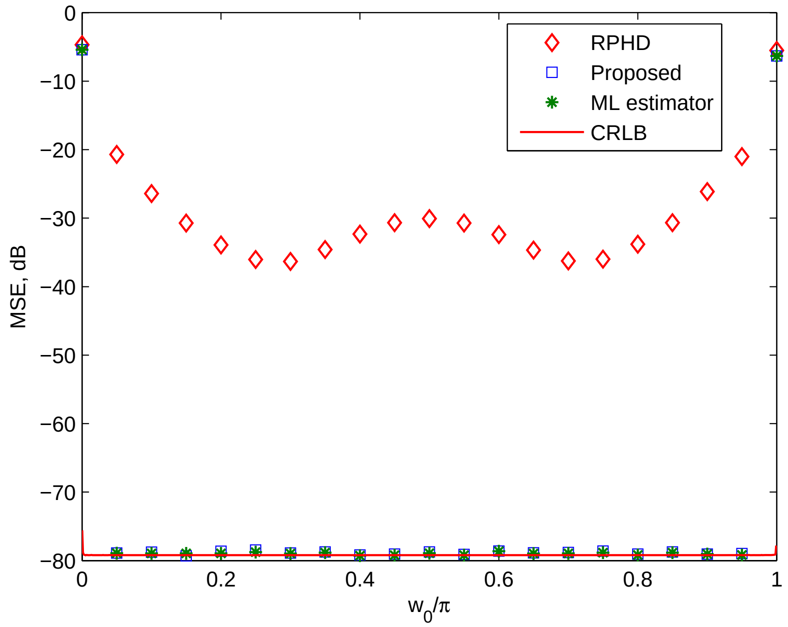

Fig. 14. Mean squared frequency errors versus ω_0, SNR=0 dB, N=1000.

APPENDIX

For notatinal convenience, let $\bar{H}(z^{-1}) = \bar{H}_{a_0,b,r}(z^{-1})$ and $G(z^{-1}) = \left[\partial \bar{H}_{a,b,r}(z^{-1})/\partial a\right]_{a=a_0}$. Specifically, one has

$$G(z^{-1}) = \frac{1}{M^3_{a_0,b,r}} \frac{B(z^{-1})}{A_b(rz^{-1})}, \tag{64}$$

where $B(z^{-1})$ is a second-order symmetric polynomial (i.e., $B(z) = B(z^{-1})z^{-2}$) given by

$$B(z^{-1}) = [2rb - a_0(1+r^2)] + 2(1 - a_0 br + b^2r^2 - r^4)z^{-1} + [2rb - a_0(1+r^2)]z^{-2}. \tag{65}$$

Introduce also the signals

$$\begin{aligned} \bar{e}\,(i; a_0) &= \underbrace{\bar{H}(z^{-1})s(i)}_{=0} + \underbrace{\bar{H}(z^{-1})u(i)}_{=\kappa(i)}, \\ \psi\,(i; a_0) &= \underbrace{G(z^{-1})s(i)}_{=x(i)} + \underbrace{G(z^{-1})u(i)}_{=v(i)}). \end{aligned}$$

Using these, we can write

$$\begin{aligned} \mathrm{E}\left[F_N^2\,(a_0)\right] &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{E}\left[e(i;a_0)\psi(i;a_0)e(j,a_0)\psi(j,a_0)\right] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{E}\left[x(i)x(j)\right] \mathrm{E}\left[\kappa(i)\kappa(j)\right] \\ &\quad + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathrm{E}\left[v(i)v(j)\kappa(i)\kappa(j)\right]. \end{aligned}$$

Since $v(i)$, $\kappa(i)$ are Gaussian, it holds that (see [18])

$$\begin{aligned}\mathrm{E}\left[v(i)v(j)\kappa(i)\kappa(j)\right] &= \mathrm{E}\left[v(i)v(j)\right]\mathrm{E}\left[\kappa(i)\kappa(j)\right] \\ &\quad + \mathrm{E}\left[v(i)\kappa(j)\right]\mathrm{E}\left[v(j)\kappa(i)\right] + \mathrm{E}\left[v(i)\kappa(i)\right]\mathrm{E}\left[v(j)\kappa(j)\right]. \qquad (66)\end{aligned}$$

Therefore we can write

$$\mathrm{E}\left[F_N^2\left(a_0\right)\right] = T_0 + T_1 + T_2 + T_3 \qquad (67)$$

where

$$T_0 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}\left[x(i)x(j)\right]\mathrm{E}\left[\kappa(i)\kappa(j)\right], \qquad (68)$$

$$T_1 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}\left[v(i)v(j)\right]\mathrm{E}\left[\kappa(i)\kappa(j)\right], \qquad (69)$$

$$T_2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}\left[v(i)\kappa(j)\right]\mathrm{E}\left[v(j)\kappa(i)\right], \qquad (70)$$

$$T_3 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{E}\left[v(i)\kappa(i)\right]\mathrm{E}\left[v(j)\kappa(j)\right] = \mathrm{E}^2\left[v(i)\kappa(i)\right]. \qquad (71)$$

Note that $E\left[v(i)\kappa(i)\right]$ is the derivative with respect to a (evaluated at $a = a_0$) of the noise gain of the normalized notch filter $\bar{H}(z^{-1})$. By design, this noise gain does not depend on a and therefore $T_3 = 0$. On the other hand, in order to evaluate T_0 we use the following approximation for large N:

$$\begin{aligned}T_0 &= \frac{1}{N^2}\sum_{l=-N+1}^{N-1}(N-|l|)\mathrm{E}\left[x(i)x(i+l)\right]\mathrm{E}\left[\kappa(i)\kappa(i+l)\right] \\ &= \frac{1}{N}\sum_{l=-N+1}^{N-1}\mathrm{E}\left[x(i)x(i+l)\right]\mathrm{E}\left[\kappa(i)\kappa(i+l)\right] - \frac{1}{N^2}\sum_{l=-N+1}^{N-1}|l|\,\mathrm{E}\left[x(i)x(i+l)\right]\mathrm{E}\left[\kappa(i)\kappa(i+l)\right] \\ &\simeq \frac{1}{N}\sum_{l=-\infty}^{\infty}\mathrm{E}\left[x(i)x(i+l)\right]\mathrm{E}\left[\kappa(i)\kappa(i+l)\right]. \qquad (72)\end{aligned}$$

Let $\{\bar{h}_k\}_{k=0}^{\infty}$ be the impulse response of $\bar{H}(z^{-1})$. Then we have

$$\begin{aligned}T_0 &\simeq \frac{\sigma_u^2}{N}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{\infty}\bar{h}_k\bar{h}_{k+l}\mathrm{E}\left[x(i)x(i+l)\right] \\ &= \frac{\sigma_u^2}{N}\mathrm{E}\left\{\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\bar{h}_k x(i-k)\bar{h}_m x(i-m)\right\} \\ &= \frac{\sigma_u^2}{N}\mathrm{E}\left\{\left[\bar{H}(z^{-1})x(i)\right]^2\right\} = 0, \qquad (73)\end{aligned}$$

since $x(i)$ is a sinusoid with frequency ω_0 and $\bar{H}(z^{-1})$ has zeros at $z = e^{\pm j\omega_0}$.

Similarly, the following approximations for T_1 and T_2 can be derived for large N:

$$T_1 \simeq \frac{\sigma_u^2}{N}\mathrm{E}\left\{\left[\bar{H}(z^{-1})G(z^{-1})u(i)\right]^2\right\}, \qquad (74)$$

$$T_2 \simeq \frac{\sigma_u^2}{N}\mathrm{E}\left\{\left[\bar{H}^2(z^{-1})u(i)\right]\cdot\left[G^2(z^{-1})u(i)\right]\right\}. \qquad (75)$$

Due to the symmetry property of the numerators of the transfer functions $\bar{H}(z^{-1})$, $G(z^{-1})$, it holds that the right-hand sides of (74) and (75) are equal; thus $T_1 \simeq T_2$. Finally, using the residue theorem, we have

$$\mathrm{E}\left[F_N^2\left(a_0\right)\right] \simeq 2T_1 \simeq \frac{2\sigma_u^4 C_{a_0,b,r}}{NM_{a_0,b,r}^8\left(1-r^2\right)\left[b^2r^2-\left(1+r^2\right)^2\right]}, \tag{76}$$

where $C_{a_0,b,r}$ is given in (50).

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