Sixth-Order Statistics-Based Non-Data-Aided SNR Estimation

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Abstract

Signal-to-noise ratio (SNR) estimation is an important task in many digital communication systems. With nonconstant modulus constellations, the performance of the classical second- and fourth-order moments estimate is known to degrade with increasing SNR. A new non-data-aided estimate is proposed, which makes use of the sixth-order moment of the received data, and which can be tuned for a particular constellation in order to extend the usable range of SNR values. The advantage of the new method is especially significant for constellations with two different amplitude levels, e.g. 16-Amplitude-and-Phase-Shift Keying (16-APSK).

Index terms: Parameter estimation, SNR estimation.

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I. Introduction

Knowledge of the signal-to-noise ratio (SNR) is a requirement in many communication systems in order to perform efficient signal detection and/or link adaptation. A number of non-data-aided (NDA) SNR estimators [1], [2] have been proposed for constant modulus (CM) constellations. Most of them, however, cannot be applied to non-CM constellations. An important exception is the so-called M_2M_4 estimator [2], [3], [4], which has been recently shown to belong to a wider family of schemes that obtain the SNR as a function of ratios of the form M_p^q/M_q^p, where M_p denotes the pth moment of the observed data [5]. An advantage of moments-based estimators is their robustness to carrier frequency and/or phase offsets.

The M_2M_4 estimator performs quite well with CM constellations: its normalized (to the true SNR) variance approaches a constant with SNR $\to \infty$, as shown in [5]. However, in the non-CM case it increases as the SNR *squared,* clearly an undesirable phenomenon. One possible approach to avoid this is to partition the set of observations in subsets corresponding to symbols of equal modulus and then perform SNR estimation only on one of these subsets [6]. However, unless the SNR is sufficiently high, many errors may occur in the partition step, with an ensuing performance loss.

We propose a novel SNR estimator that makes use of the second, fourth and sixth moments of the observations. Although its normalized variance eventually rises with increasing SNR for non-CM constellations, the range of SNR values with acceptable performance is larger than that of the M_2M_4 estimator when the new scheme is properly tuned. In particular, for sources with two distinct amplitude levels, the rate of increase is linear (rather than quadratic) in the SNR.

II. Moments-based SNR estimation

Assuming a quasistatic flat-fading channel model, the sampled (at the symbol rate) matched filter output is given by

$$r_k = \sqrt{S} \cdot x_k + n_k, \quad k = 1, \dots, K, \tag{1}$$

where x_k are the complex-valued transmitted symbols, $\sqrt{S}$ is the unknown channel gain, and the complex-valued noise samples n_k are independent, zero-mean and circular Gaussian with variance N. The constellation moments are denoted as $c_p \doteq E\{|x_k|^p\}$; energy normalization is assumed, i.e. $c_2 = 1$.

Given the K samples $\{r_k\}$, the goal is to estimate the SNR defined as $\rho \doteq S/N$, or equivalently the *normalized SNR* $z \doteq S/(S+N) = \rho/(1+\rho)$; note that $z \in (0,1)$. Moments-based SNR estimators are functions of the sample moments

$$\hat{M}_p \doteq \frac{1}{K}\sum_{k=1}^{K} |r_k|^p, \tag{2}$$

which in turn are unbiased and consistent estimators of the true moments $M_p \doteq E\{|r_k|^p\}$. Using the fact that $E\{|n_k|^{2m}\} = m! \cdot N^m$, the even-order moments M_{2n} are seen to admit closed form expressions in terms of S, N, and c_{2m}, $0 \leq m \leq n$:

$$M_{2n} = \sum_{m=0}^{n} \frac{(n!)^2}{(n-m)!\,(m!)^2} c_{2m} S^m N^{n-m} \tag{3}$$

For example, one has

$$M_2 = S + N, \qquad M_4 = c_4 S^2 + 4SN + 2N^2, \tag{4}$$

and thus $2M_2^2 - M_4 = (2-c_4)S^2$, or $2 - (M_4/M_2^2) = (2-c_4)z^2$. This results in the M_2M_4 estimate

$$\hat{z} = \sqrt{\frac{2 - \hat{M}_4/\hat{M}_2^2}{2 - c_4}}, \qquad \hat{\rho} = \frac{\hat{z}}{1-\hat{z}}. \tag{5}$$

III. A NEW ESTIMATE USING SIXTH-ORDER STATISTICS

The M_2M_4 estimate is obtained via a combination of moments that depends only on the normalized SNR z. Carrying this idea one step further, one can write

$$M_6 = c_6 S^3 + 9c_4 S^2 N + 18SN^2 + 6N^3, \tag{6}$$

$$M_2^3 = S^3 + 3S^2N + 3SN^2 + N^3, \tag{7}$$

$$M_2 M_4 = c_4 S^3 + (4 + c_4)S^2 N + 6SN^2 + 2N^3. \tag{8}$$

We seek a linear combination of (6)-(8) in which as many N-dependent terms as possible should be removed. Thus, let $D \doteq M_6 - aM_2^3 - bM_2M_4$. We note that by choosing $a = 2(3-b)$, the terms in SN^2 and N^3 cancel out, yielding

$$\begin{aligned} D &= M_6 - 2(3-b)M_2^3 - bM_2M_4 \\ &= [(c_6 - 6) - b(c_4 - 2)]S^3 + (9-b)(c_4-2)S^2N. \end{aligned} \tag{9}$$

Now we substitute $N = M_2 - S$ and divide both sides of (9) by M_2^3 to obtain

$$\frac{D}{M_2^3} = (c_6 - 9c_4 + 12)z^3 + (9-b)(c_4-2)z^2. \tag{10}$$

Hence, with $\hat{D} \doteq \hat{M}_6 - 2(3-b)\hat{M}_2^3 - b\hat{M}_2\hat{M}_4$, an estimate $\hat{z}$ can be obtained by solving for the root in $(0,1)$ of

$$(c_6 - 9c_4 + 12)\hat{z}^3 + (9-b)(c_4-2)\hat{z}^2 - \frac{\hat{D}}{\hat{M}_2^3} = 0. \tag{11}$$

For a given b and a given constellation, this root can be tabulated in terms of $\hat{D}/\hat{M}_2^3$. Alternatively, the following iterative rule will find such root in a few steps:

$$\hat{z}^{(n+1)} = \sqrt{\frac{\hat{D}/\hat{M}_2^3}{\alpha\hat{z}^{(n)} + \beta}}, \tag{12}$$

where $\alpha \doteq c_6 - 9c_4 + 12$ and $\beta \doteq (9-b)(c_4-2)$. Either $\hat{z}^{(0)} = 0$ or 1 can be used as starting point.

Observe that for large $|b|$ the reference statistic satisfies $\hat{D}/\hat{M}_2^3 \approx b(2 - \hat{M}_4/\hat{M}_2^2)$, whereas the solutions of (11) approximately satisfy $b(2-c_4)\hat{z}^2 - \hat{D}/\hat{M}_2^3 \approx 0$. From these, it follows that the proposed estimator will approach the M_2M_4 estimator (5) for sufficiently large $|b|$. On the other hand, if $b = 0$ then $D/M_2^3 = M_6/M_2^3 - 6$, and the resulting estimator is a particular case ($k = 6$, $l = 2$) of the family proposed in [5]. The choice of the free parameter b should be tailored to the particular constellation, as the variance analysis will show.

IV. Variance analysis

Since $\hat{\rho} = \hat{z}/(1-\hat{z})$, we can write $\hat{\rho} = f(\hat{\boldsymbol{m}})$, where $\hat{\boldsymbol{m}} \doteq [\hat{M}_2\ \hat{M}_4\ \hat{M}_6]^T$, $f(\hat{\boldsymbol{m}}) \doteq g(\hat{\boldsymbol{m}})/(1-g(\hat{\boldsymbol{m}}))$, and the function g is implicitly given by

$$\alpha g^3 + \beta g^2 - h = 0, \quad h(\hat{\boldsymbol{m}}) \doteq \frac{\hat{D}}{\hat{M}_2^3} = \frac{\hat{M}_6}{\hat{M}_2^3} - b\frac{\hat{M}_4}{\hat{M}_2^2} - 2(3-b). \tag{13}$$

Let $\boldsymbol{m} \doteq [M_2\ M_4\ M_6]^T$ be the vector of the true moments. A first-order Taylor expansion of f about the point $\hat{\boldsymbol{m}} = \boldsymbol{m}$ yields the small error approximation $\hat{\rho} \approx \rho + \boldsymbol{v}^T(\hat{\boldsymbol{m}} - \boldsymbol{m})$, where $\boldsymbol{v} \doteq \nabla f|_{\hat{\boldsymbol{m}}=\boldsymbol{m}}$. Hence the estimation variance is approximately given by $\mathsf{Var}\{\hat{\rho}\} \approx \boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v}$, where $\boldsymbol{C}$ is the covariance matrix of $\hat{\boldsymbol{m}}$, with elements $\boldsymbol{C}_{ij} = (M_{2(i+j)} - M_{2i}M_{2j})/K$, for $i, j \in \{1,2,3\}$.

Note that $\nabla f = (1-g)^{-2} \cdot \nabla g$. Using this, and taking partial derivatives in (13), one finds that

$$\nabla g = -\frac{1}{3\alpha g^2 + 2\beta g}\nabla h \quad \Rightarrow \quad \boldsymbol{v} = -\frac{(1+\rho)^4}{(3\alpha + 2\beta)\rho^2 + 2\beta\rho} \times \frac{1}{M_2^4}\begin{bmatrix} 3M_6 - 2bM_2M_4 \\ bM_2^2 \\ -M_2 \end{bmatrix}. \tag{14}$$

With this, it is straightforward but tedious to compute the approximate variance $\boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v}$. It turns out that $\boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v}$ depends only on the *normalized moments* M_{2n}/M_2^n, $1 \leq n \leq 6$, which are seen from (3) to be functions of S and N through ρ only. The final expression of the variance is found to be

$$\mathsf{Var}\{\hat{\rho}\} \approx \frac{1}{K} \cdot \frac{A_8\rho^8 + A_7\rho^7 + \cdots + A_1\rho + A_0}{B_4\rho^4 + B_3\rho^3 + B_2\rho^2}, \tag{15}$$

where A_i, B_j are constants depending on b and the constellation moments only. In particular, one has

$$B_4 = (\zeta_1 b + \zeta_0)^2,\ B_3 = 2\zeta_1(b-9)(\zeta_1 b + \zeta_0),\ B_2 = \zeta_1^2(b-9)^2, \tag{16}$$

where $\zeta_0 \doteq 3(3c_4 - c_6)$ and $\zeta_1 \doteq 2(c_4 - 2)$. The numerator coeficients are also quadratic in b:

$$A_m = \kappa_2^{(m)} b^2 + \kappa_1^{(m)} b + \kappa_0^{(m)}, \qquad m = 0, 1, \ldots, 8, \tag{17}$$

with $\kappa_i^{(m)}$ polynomial functions of c_4, c_6, $\ldots$, c_{12}.

A. CM constellations

For CM sources, $c_p = 1$ for all p, and one finds that $A_8 = A_7 = 0$ for all b, whereas $A_6 = 8(b-3)^2$, $A_5 = 48(b^2 - 9b + 19)$. On the other hand, $B_4 = 4(b-3)^2$, $B_3 = 8(b-9)(b-3)$ and $B_2 = 4(b-9)^2$. Note that for $b = 3$, $B_4 = B_3 = 0$ but $A_5 \neq 0$, $B_2 \neq 0$. Thus, as long as $b \neq 3$, the variance of the proposed estimator is $\mathcal{O}(\rho^2)$, with leading coefficient $A_6/B_4 = 2$ (it will be $\mathcal{O}(\rho^3)$ for $b = 3$). In addition b should be chosen outside the interval $(3, 9)$ or otherwise the variance will exhibit a sharp peak near $\rho = (9-b)/(b-3)$, the positive root of the denominator of (15). For $b < 3$ or $b \geq 9$ the behavior is quite similar to that of the M_2M_4 estimator for CM signals, derived in [5] (recall that for large $|b|$ the proposed method approaches the M_2M_4 estimator).

B. Non-CM constellations

The coefficients determining A_8 are found to be

$$\kappa_0^{(8)} \doteq c_{12} - 4c_6^2 + 9c_4c_6^2 - 6c_6c_8, \tag{18}$$

$$\kappa_1^{(8)} \doteq -2c_{10} + 4c_4c_8 + 6c_6^2 - 12c_4^2c_6 + 4c_4c_6, \tag{19}$$

$$\kappa_2^{(8)} \doteq c_8 - 4c_4c_6 + 4c_4^3 - c_4^2. \tag{20}$$

From (16), $B_4 = 0$ implies $B_3 = 0$ as well. In that case, if $A_8 \neq 0$, then the variance is $\mathcal{O}(\rho^6)$. On the other hand, if $B_4 \neq 0$ then the variance is $\mathcal{O}(\rho^4)$ in general, similarly to that of the M_2M_4 estimator [5]. In order to extend as much as possible the usable SNR range of the new estimator, b can be tuned for a given constellation. We propose to pick b to minimize the leading coefficient A_8/B_4. This is achieved for

$$b = b_\star \doteq \frac{2\zeta_1\kappa_0^{(8)} - \zeta_0\kappa_1^{(8)}}{2\zeta_0\kappa_2^{(8)} - \zeta_1\kappa_1^{(8)}}. \tag{21}$$

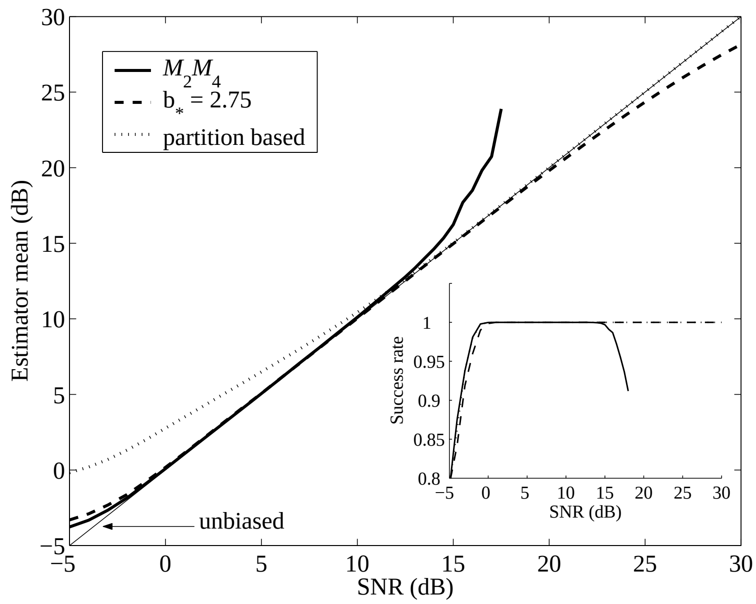

Fig. 1. Mean value and sucess rate of several SNR estimators for $K = 1000$ and 16-APSK modulation.

Unfortunately, the equation $\kappa_2^{(8)}b^2 + \kappa_1^{(8)}b + \kappa_0^{(8)} = 0$ has no real roots in general, meaning that A_8 cannot be made zero. However, a special situation is found when the constellation has only two different amplitude levels, R_1 and R_2, with associated probabilities p and $1-p$ respectively. Then

$$c_{2n} = \frac{p + (1-p)w^{2n}}{(p + (1-p)w^2)^n}, \tag{22}$$

with $w \doteq R_2/R_1$ the ring ratio. Using (22) one can show that $(\kappa_1^{(8)})^2 - 4\kappa_2^{(8)}\kappa_0^{(8)} = 0$, so that $A_8 = \kappa_2^{(8)}(b - b_\star)^2$, where $b_\star$ given by (21) simplifies to $b_\star = -\kappa_1^{(8)}/(2\kappa_2^{(8)})$. This means that for these constellations, A_8 can be made zero via (21), so that the variance will be $\mathcal{O}(\rho^3)$. This constitutes a sizable improvement over the M_2M_4 estimate, whose variance is $\mathcal{O}(\rho^4)$. In terms of p, w, the optimum value $b_\star$ is given by

$$b_\star = \frac{2(1-p)w^4 + (1-2p)w^2 - 2p}{(1-p)^2w^4 - p^2}. \tag{23}$$

V. Simulation results

We show the results obtained via Monte Carlo simulation (10^4 runs for each point) for a 16-APSK constellation such as those specified in the DVB-S2 standard [7], consisting of an inner and an outer ring with 4 and 12 symbols respectively. DVB-S2 specifies six possible ring ratios for 16-APSK, ranging from $w = 2.57$ to $w = 3.15$ depending on the code rate.

Fig. 1 shows the mean value of the M_2M_4, the newly proposed, and the partition based [6] estimators when $K = 1000$ samples and $w = 3.15$. Also shown is the *success rate* of each method, estimated as the fraction of Monte Carlo trials for which the algorithm did not run into a negative radicand[1] (see (5) and (12)). The M_2M_4 scheme is seen to be useful for SNR < 15 dB only. The partition based method, on the other hand, is severely biased for SNR < 10 dB due to the increasing number of decision errors when partitioning the observation data set as the SNR decreases. The proposed method uses an optimal value $b_\star = 2.75$, extending the usable range of the M_2M_4 scheme into higher SNR values without the large bias observed for the partition based method for low SNR.

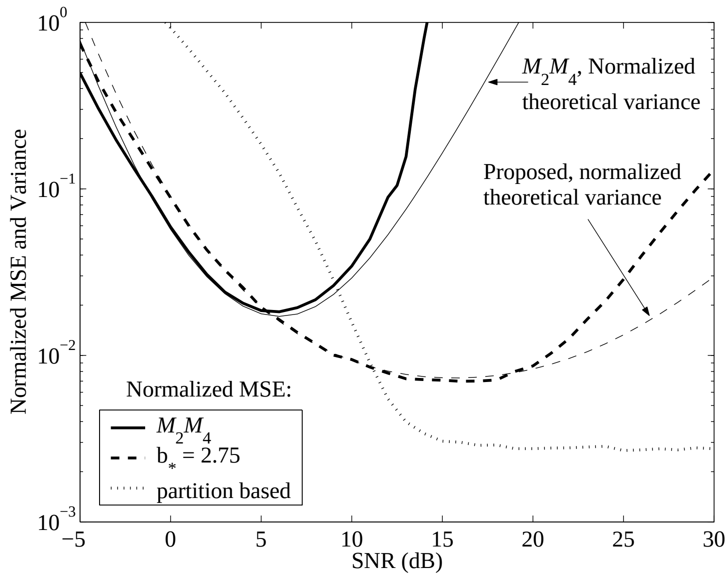

Fig. 2. MSE and theoretical variance (normalized to true SNR) of several SNR estimators for $K = 1000$ and 16-APSK modulation.

Fig. 2 shows the MSE of the three schemes, normalized to the true SNR. The new estimator is slightly worse than the M_2M_4 scheme below 5 dB, but the improvement for high SNR is evident. The linearized normalized variances for the M_2M_4 (from [5]) and the proposed method (from (15)) are in good agreement with the simulation results. For high SNR both methods become noticeably biased, with the corresponding MSE increase with respect to the theoretical variance.

[1]Note that the probability of a negative radicand for a given realization is always nonzero due to Gaussianity of the noise.

VI. Conclusion

A new NDA SNR estimator has been proposed which extends the usable range of the well-known M_2M_4 estimator for non-CM constellations, particularly for two-amplitude-level sources. It is based on the moments of the observations and therefore can be applied before carrier recovery has been established. Complexity of the new method is about 33% higher than that of the M_2M_4 scheme, due to the need to compute an additional moment. A topic for further research is whether performance can be further improved by including other higher-order moments (e.g. M_8) in the estimator.

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