# Bandwidth Allocation in Partial Duplex Relaying

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***Abstract*—The use of relays to improve the performance of wireless systems finds some limitations, which in the case of Amplify and Forward repeaters come from the need to decouple the incoming signal from its outgoing amplified version. In this paper we analyze the performance of Amplify and Forward relays for a partial overlap between the input and output frequency contents. By considering the relay as a Linearly Periodic Time Varying System we compute its transmission capacity, which depends strongly on the undesired coupling from the amplified output signal into the input antenna. As a byproduct, a fair comparison between half-duplex and full-duplex operation is obtained, complementing some initial results which can be found in the literature.**

## I. Introduction

Relaying is a widespread resource to extend the coverage of wireless links by appropriate amplification of the input signal [1]. The most simple mechanism to avoid the contamination of the input by the output signal of the relay is the use of different frequency bands or time slots; this duplexing mechanism is not efficient in terms of spectral efficiency, due to the doubling of spectral or time resources with respect to a unrelayed direct link. We will refer to this scheme as Half-Duplex (HD) operation, as opposed to its Full-Duplex (FD) counterpart, for which input and output signals use simultaneously the same portion of the spectrum. FD is in place in some practical settings, and usually requires some sort of echo cancellation processing to attenuate the coupled self-interference. See, for example [2], where the operation of practical on-frequency gap-fillers for terrestrial television broadcasting is studied. In [3] a comparison between the capacity of FD and HD is made for different coupling levels. The coupling can be caused by the residual echo due a non-perfect cancellation, or by the loopback signal in absence of any active cancellation scheme. The range for which FD outperforms HD in [3] is quite limited despite the additional bandwidth, since the self-interference is assimilated to noise in a sort of pessimistic analysis. In [4] a hybrid scheme switches between HD and FD operation as a function of the capacity of the source to relay and relay to destination links. A three phase scheduling allocates time for source to relay, relay to destination, and simultaneous transmission and reception by the relay; this combination of orthogonal and simultaneous operation requires the knowledge of the capacity of the source-relay and relay-destination links in both modes. An opportunistic switching between FD and HD is also considered in [5].

In this work we measure the relay performance by evaluating its transmission capacity as a function of the allocation of the overall bandwidth to the input and the output; as particular instances, HD and FD are also included. Since the study will focus on the relay itself, the source-relay and relay-destination channels will be considered as non-selective, and noise will be present only at the final receiver.

## II. Partial Duplex Relay

Figure 1 depicts the operations undergone by a signal $x(t)$ through a generalized Amplify and Forward (AF) relay. An input signal $x(t)$ occupying a bandwidth $B_u$ is filtered, shifted in frequency by $f_0 = B - B_u$, and finally amplified[1]. The potential coupling from the output to the input between mutually present frequencies (if $B_u > B/2$) is the reason to include a feedback branch. The filter $L_u(f)$ magnitude response is given by

$$|L_u(f)| = \begin{cases} 1 & 0 \le f \le B_u \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The filter delays also the signal by $t_0$, so $L_u(f) = |L_u(f)|e^{-j(2\pi f t_0 + \theta_0)}$, where a phase $\theta_0$ is included for generalization. The output signal $y(t)$ will be assumed to be contaminated by noise before reaching the final receiver, although no additional channel impairment is added. No direct link from the transmitter to the receiver will be assumed.

The design parameter is the bandwidth allocated to the input and the output signals $B_u = B/2 + \Delta B$. Under the AF configuration in Figure 1, $\Delta B = 0$ corresponds to a half-duplex (HD) relay, with input and ouptut spectra fully decoupled, whereas $\Delta B = B/2$ represents the full-duplex (FD) relay, for which input and output bandwidths overlap entirely. All cases in between will be labeled as **Partial Duplex (PD)**.

The operations performed by the AF relay can be written in the Fourier domain as

$$Y(f) = \sqrt{g}X(f-f_0)L_u(f-f_0) + \sqrt{\alpha g}Y(f-f_0)L_u(f-f_0), \tag{2}$$

[1] We do not include possible non-linear effects in this study.

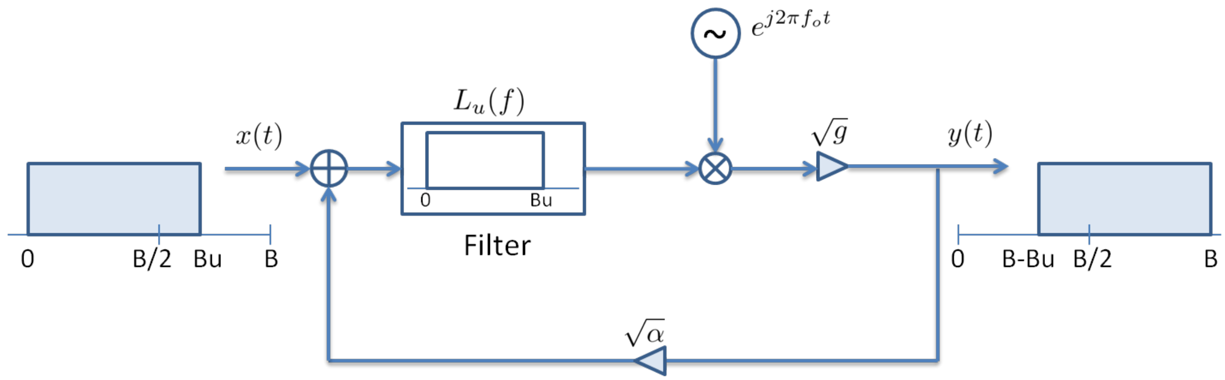



Fig. 1: Functional description of partial duplex relay. A practical implementation is expected to include an initial downconversion stage complemented by an upconversion to the output frequency band. The difference between the corresponding oscillator frequencies is given by $f_0$.

a recursive relation which can be unfolded to yield

$$Y(f) = \sqrt{g}X(f-f_0)L_u(f-f_0)+ \overbrace{\sqrt{g}\sum_{k=1}^{K_{max}}(\sqrt{\alpha g})^k X(f-(k+1)f_0)\Pi_{m=1}^{k+1}L_u(f-mf_0)}^{\text{self-interference}}. \tag{3}$$

The number of terms contributing to the self-intererence sum is finite except for $\Delta B = B/2$ (FD case), and it is given by

$$K_{max} = \left\lceil \frac{B/2+\Delta B}{B/2-\Delta B} \right\rceil - 1 \tag{4}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. This bound can be readily obtained from the bandwidth of the filter $L_u(f)$, $B/2+\Delta B$, and the frequency shift $f_0 = B/2 - \Delta B$ applied to the main path in Figure 1.

The signal output power $P_y$ is fixed due to the presence of gain controls. The amplification gain $g$ in Figure 1 is not set beforehand, but rather is adjusted to achieve the prescribed output power considering the presence of feedback. If the coupling gain $\alpha = 0$ is zero, then the amplifier gain is simply written as $g = P_y/P_x$, with $P_x$ denoting the input power.

As performance metric we will use the channel capacity from the AF relay to the final destination, provided that there is not direct link from the source. The noise generated by the AF relay should also be considered, although we will assume that its relative contribution is much less significant than that of the final receiver noise. A reference signal to noise ratio (snr) is defined for flat spectrum noise with power spectral density $N_o$ as

$$\text{snr} \triangleq P_y/N_oB. \tag{5}$$

If self-interference is assimilated to noise, then a fast degradation of FD performance with the **loop gain** $\text{LG} = \alpha P_y/P_x$ occurs as a result. In such a case, the interfering output power can be readily seen to be equal to $P_y - gP_x$, whose contribution for fixed $P_y$ and $P_x$ grows with LG, since the gain $g$ gets smaller. In consequence, the consideration of self-interference as noise leads to overly pessimistic results. For a more accurate analysis, we can note that for $B_u < B$ the relay cannot be considered as a Linear Time Invariant (LTI) system, but rather as a Linearly Periodic Time Varying (LPTV) system. We will use this consideration in the next section to obtain the PD relay capacity, together with an approximation for additional insight.

## III. Capacity of PD Relay

The input-output relationship of the PD relay can be expressed as a linear time variant filter:

$$r(t) = y(t) + w(t) = \sqrt{g}\int_{\infty}^{\infty} p(t,\tau)x(t-\tau)d\tau + w(t). \tag{6}$$

This is the time counterpart of (3), with the noise at the final receiver explicitely added, and the filter response $p(t,\tau)$ given by

$$p(t,\tau) = \sum_{k=1}^{K_{max}+1}(\sqrt{g\alpha})^{k-1}\cdot \left(l_u(\tau)e^{j2\pi\tau/T_0} * \ldots * l_u(\tau)e^{j2\pi k\tau/T_0}\right)e^{j2\pi k(t-\tau)/T_0} \tag{7}$$

with $T_0 = 1/f_0$ and $K_{max}$ defined in (4). Note that the input-output relation corresponds to an LPTV sytem, since $p(t,\tau) = p(t+T_0,\tau)$. In discrete-time form, if the sampling rate is equal to $1/T$, we have

$$y(nT) = \sqrt{g}\sum_{m=-\infty}^{\infty} p(nT,mT)x((n-m)T) \tag{8}$$

with the time-variant discrete-time impulse response

$$p(nT,mT) = \sum_{k=1}^{K_{max}+1}(\sqrt{g\alpha})^{k-1}\cdot \left(l_u(mT)e^{j2\pi mT/T_0} * \ldots * l_u(mT)e^{j2\pi mkT/T_0}\right)e^{j2\pi(n-m)kT/T_0}. \tag{9}$$

After defining

$$l_k(mT) \triangleq l_u(mT)e^{j2\pi mT/T_0} * \ldots * l_u(mT)e^{j2\pi mkT/T_0} \quad (10)$$

(9) reads as

$$p(nT, mT) = \sum_{k=1}^{K_{max}+1} (\sqrt{g\alpha})^{k-1} l_k(mT) e^{-j2\pi mkT/T_0} e^{j2\pi nkT/T_0}. \quad (11)$$

If we choose $T = 1/B$, then the Nyquist criterion is satisfied for all bandwidths under consideration in Figure 1, and the period $N_{ch}$ of the discrete-time impulse response, such that $p((n + N_{ch})T, mT) = p(nT, mT)$, is given by

$$N_{ch} \triangleq \frac{T_0}{T} = \frac{B}{B - B_u} \quad (12)$$

which for the range of considered bandwidths is such that $2 \leq N_{ch} < \infty$. Although $N_{ch}$ is not necessarily integer, the subsequent analysis is simpler by doing this assumption; the simulations will illustrate how the approximation in Section III-A allows to extend the results to any value of $T_0/T$.

The relay LPTV response is rewritten now as

$$p_n(mT) \triangleq p(nT, mT) = \sum_{k=1}^{K_{max}+1} (\sqrt{g\alpha})^{k-1} l_k(mT) e^{-j2\pi mk/N_{ch}} e^{j2\pi nk/N_{ch}}. \quad (13)$$

If the discrete-time equivalent response of the filter in the PD relay goes from 0 to $\ell$, then the time duration of $p_n(mT)$ is $\ell_p + 1$, with $\ell_p \triangleq (K_{max} + 1)\ell$ for all $n$.

Somewhat surprisingly, the capacity computation of LPTV channels has not been directly addressed in the literature till recently. The results exposed next follow the derivation in [6], applied to Power Line Communications, and based on the assimilation of the channel to a MIMO LTI system. Ideas from [7] can also be exploited; in the latter the capacity of the multivariate Gaussian channel with memory is obtained, by formulating the input-output relationship as a MIMO channel with memory, although the term MIMO was not used at that time.

The relay channel capacity can be derived by transforming the original scalar model into a vector model. The size $M$ of the input block is given by

$$M = \left( \left\lceil \frac{\ell_p + 1}{N_{ch}} \right\rceil + \kappa \right) N_{ch} \quad (14)$$

whereas the dimension of the output vector is $M - \ell_p$. The integer $\kappa$ is such that $\kappa > 0$, and used to determine the number of periods included in the input block. The input-output relationship can be expressed in matrix form as

$$\mathbf{r}[n] = \sqrt{g}\mathbf{P}\mathbf{x}[n] + \mathbf{w}[n], \quad (15)$$

with $\mathbf{x}[n] = [x(nMT), \ldots, x((nM + M - 1)T)]^t$, $\mathbf{r}[n] = [r((nM + \ell_p)T), \ldots, r((nM + M - 1)T)]^t$, the noise $\mathbf{w}[n] = [w((nM + \ell_p)T), \ldots, w((nM + M - 1)T)]^t$, and the $(M - \ell_p) \times M$ channel matrix given by

$$\mathbf{P} \triangleq \begin{pmatrix} p_{\ell_p}(\ell_p T) & \ldots & p_{\ell_p}(0) & \ldots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \ldots & p_{M-1}(\ell_p T) & \ldots & p_{M-1}(0) \end{pmatrix}. \quad (16)$$

Note that the dimension of the output vector, $M - \ell_p$, is lower than the $M$ samples of the input block. Nevertheless, the impact on the capacity derivation decreases as $M$ grows (by increasing $\kappa$), and the true capacity can be obtained as the limit $\lim_{M\to\infty} C_M$ [6], with $C_M$ denoting the achievable rate of the truncated MIMO model with channel matrix $\mathbf{P}$ in (16) for a given input covariance $\mathbf{C}_x \triangleq \mathbb{E}\left[\mathbf{x}[n]\mathbf{x}^H[n]\right]$:

$$C_M/B = \frac{1}{M - \ell_p} \log_2 \det \left( \mathbf{I} + \frac{g}{N_0 B} \mathbf{P}\mathbf{C}_x\mathbf{P}^H \right). \quad (17)$$

Since we are considering non-frequency selective channels, we will assume that the frequency content is flat in the occupied bandwidth:

$$C_x(mT) = P_x \cdot \text{sinc}\left(mB_u/B\right) e^{j\pi m B_u/B}. \quad (18)$$

Finally, for output power $P_y$, the PD relay gain is computed as

$$g = \frac{P_y}{\text{trace}\{\mathbf{P}\mathbf{C}_x\mathbf{P}^H\}/(M - \ell_p)}. \quad (19)$$

### A. Frequency-domain approximation

The time domain relation between input and output vectors in (15) can equivalently be expressed in the frequency domain. If $\mathbf{F}$ denotes the Discrete Fourier Transform (DFT) matrix with the appropriate dimension, then $\mathbf{r}[n] = \mathbf{F}^H\mathbf{R}[n]$, $\mathbf{x}[n] = \mathbf{F}^H\mathbf{X}[n]$, and the relay operation reads as

$$\mathbf{R}[n] = \mathbf{F}\mathbf{P}\mathbf{F}^H\mathbf{X}[n] + \mathbf{W}[n] \quad (20)$$

with $\mathbf{R}[n]$, $\mathbf{X}[n]$ and $\mathbf{W}[n]$ the DFTs of the $n$th output, input and noise blocks, respectively. The corresponding continuous Fourier Transform relation in (3) can be used as an approximation. We consider $L$ carriers, with intercarrier spacing given by $\Delta f = B/L$, and the frequency offset in Figure 1 equal to $f_0 = (L - N)\Delta f$ if $B_u = \Delta f \cdot N$. Inter Carrier Interference (ICI) will be a consequence of the coupling of carriers which have altered their positions after passing through the relay. The relation between $\mathbf{R}[n]$ and $\mathbf{X}[n]$ follows from (3), and can be expressed in matrix form as

$$\mathbf{R}[n] = \mathbf{H}\mathbf{X}[n] + \mathbf{W}[n], \quad (21)$$

with $\mathbf{H}$ an $L \times L$ matrix, and $\mathbf{R}[n], \mathbf{X}[n], \mathbf{W}[n]$ $L \times 1$ vectors. Note that, in the noiseless case, input and output vectors have $N - L$ zeros:

$$\begin{pmatrix} 0 \\ \cdot \\ 0 \\ \square \\ \vdots \\ \square \end{pmatrix} = \left( \begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \sqrt{g} \cdot \mathbf{T} \cdot \mathbf{\Lambda} & \mathbf{0} \end{array} \right) \begin{pmatrix} \square \\ \vdots \\ \square \\ 0 \\ \cdot \\ 0 \end{pmatrix}. \quad (22)$$

where $\mathbf{T}$ is lower triangular and $\mathbf{\Lambda}$ diagonal. Their corresponding elements are given by

$$[\mathbf{\Lambda}]_{nn} = e^{j\theta_0} e^{-j2\pi n\Delta f t_0},\ n = 1, \ldots, N, \quad (23)$$

$$[\mathbf{T}]_{nn} = 1,\ n = 1, \ldots, N, \quad (24)$$

$$[\mathbf{T}]_{nm} = 0,\ m > n, \quad (25)$$

$$[\mathbf{T}]_{nm} = 0,\ n - m \neq k(L-N), k \text{ integer}, \quad (26)$$

$$[\mathbf{T}]_{nm} = (\sqrt{g\alpha})^k e^{jk\theta} e^{-j2\pi n\Delta f t_0 k} e^{j2\pi t_0 \Delta f (L-N)k(k+1)/2} \cdot |L_u((n - k(L-N))\Delta f)|,\ n - m = k(L-N). \quad (27)$$

The last expression applies for $1 \leq k \leq K_{max}$, with

$$K_{max} = \left\lceil \frac{N}{L-N} \right\rceil - 1. \quad (28)$$

For illustration purposes, the magnitude of the $N \times N$ matrix $\mathbf{T}$ can be seen to be the following for $L - N = 2$ and an ideal filter response $L_U(f)$:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\ \sqrt{g\alpha} & 0 & 1 & 0 & \ldots & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & \sqrt{g\alpha} & 0 & 1 & 0 & \ldots & 0 \\ (\sqrt{g\alpha})^k & 0 & \ldots & \sqrt{g\alpha} & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ (\sqrt{g\alpha})^{K_{max}} & 0 & (\sqrt{g\alpha})^k & \ldots & 0 & \sqrt{g\alpha} & 0 & 1 \end{pmatrix}. \quad (29)$$

The sum rate of all carriers is bounded by the relay channel capacity, which from (21) give the following achievable rate for flat-spectrum inputs:

$$C/B = \frac{1}{L} \log_2 \det \left( \mathbf{I} + \text{snr} \frac{L}{\text{trace}\{\mathbf{T}\mathbf{T}^H\}} \mathbf{T}\mathbf{T}^H \right). \quad (30)$$

Note that a potential receiver extracting the information from the PD relayed signal would require higher complexity due to the time-varying behavior, translated into the interfering terms depicted in (22).

In practice, the intensity of the self-coupling is usually measured as the relative magnitude of the fedback signal with respect to the input signal, parameterized by the loop gain LG introduced in Section II. For a fair and realistic comparison, the output power $P_y$ will be fixed by the appropriate gain control; from (22), this power is given by

$$P_y = g P_x \frac{1}{N} \text{trace}\{\mathbf{T}\mathbf{T}^H\}, \quad (31)$$

from which follows

$$g\alpha = \frac{\alpha P_y}{P_x} \frac{1}{\frac{1}{N}\text{trace}\{\mathbf{T}\mathbf{T}^H\}} = \text{LG} \frac{1}{\frac{1}{N}\text{trace}\{\mathbf{T}\mathbf{T}^H\}}. \quad (32)$$

From the expression of $\mathbf{T}$, we have that $\text{trace}\{\mathbf{T}\mathbf{T}^H\} = \sum_{k=0}^{K_{max}} (g\alpha)^k (N - k(L-N))$, so that $g\alpha$ is the real solution of the equation $\sum_{k=0}^{K_{max}} (g\alpha)^{k+1} (N - k(L-N)) - N \cdot \text{LG} = 0$.

## IV. Numerical Results

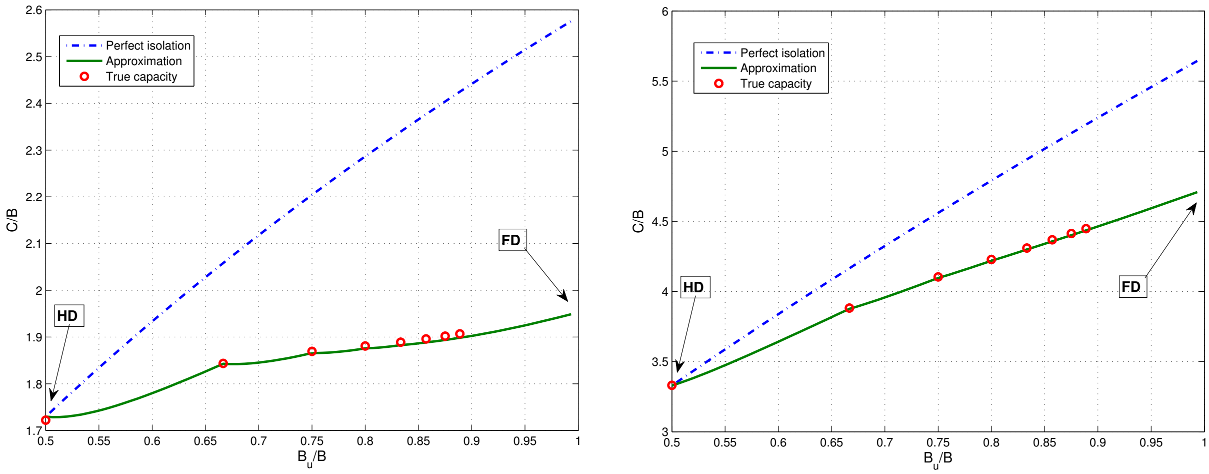
We have computed the PD maximum spectral efficiency $C/B$ in (17) and its approximation (30), in both cases as the relay achievable rate considering uniform power allocation across the input bandwidth $B_u$. The true capacity is computed for $M$ large enough and $B_u = B(1 - 1/r)$, with $r$ an integer value going from 2 to 9; it can be checked that the period $N_{ch}$ in (12) is directly equal to $r$. The operation point is determined by the bandwidth $B_u/B$, the signal-to-noise ratio and the magnitude of the coupling LG. As reference we plot also the spectral efficiency for $LG = -\infty$ dBs, given by $C/B = \frac{B_u}{B} \log_2(1 + P_y/(N_0 B_u))$, and labeled as *perfect isolation*. The spectral efficiency is plotted in Figures 2 and 3 for two values of the snr, 7 and 17 dB. The filter $L_u(f)$ was implemented as a truncated sinc such that $t_0(B - B_u) = 5$. On the other side, it can be proved that the approximation (30) is independent of the delay $t_0$. Note the similarity between the true capacity values and those obtained with the frequency domain approach, which does not rely on the existence of a periodic behavior of the relay and can be computed more easily. Since the true achievable rate has been computed only for values of the form $B_u = B(1 - 1/r)$, it is a question for further research to learn if the non-smooth behavior of the rate with $B_u$, predicted by the approximation (30), applies also to the true values. From the tests that we have performed, we can conclude that low values of $L$, lower than 100, are enough to predict accurately the spectral efficiency of the PD relay. As expected, the performance degrades with the loop gain LG, although it is remarkable that FD beats HD even for LG = 0 dB, that is, for an echo which has the same power as the input signal.

## V. Conclusions

An AF relay with partial overlapping between input and output spectra has been proposed and analyzed in terms of its achievable rate. The behavior for different overlapping degrees has been obtained by exploiting the LPTV nature of this PD relay. A frequency domain approach has been also presented which shows a remarkable accuracy to predict the true capacity. As a by-product, HD and FD operation can be compared within this framework; the most favorable mode is a function of the relay self-coupling and signal to noise ratio. It turns out that FD can beat HD even for a low isolation from the output coupling, if the contribution of the feedback is properly accounted, and not simply disregarded as noise.

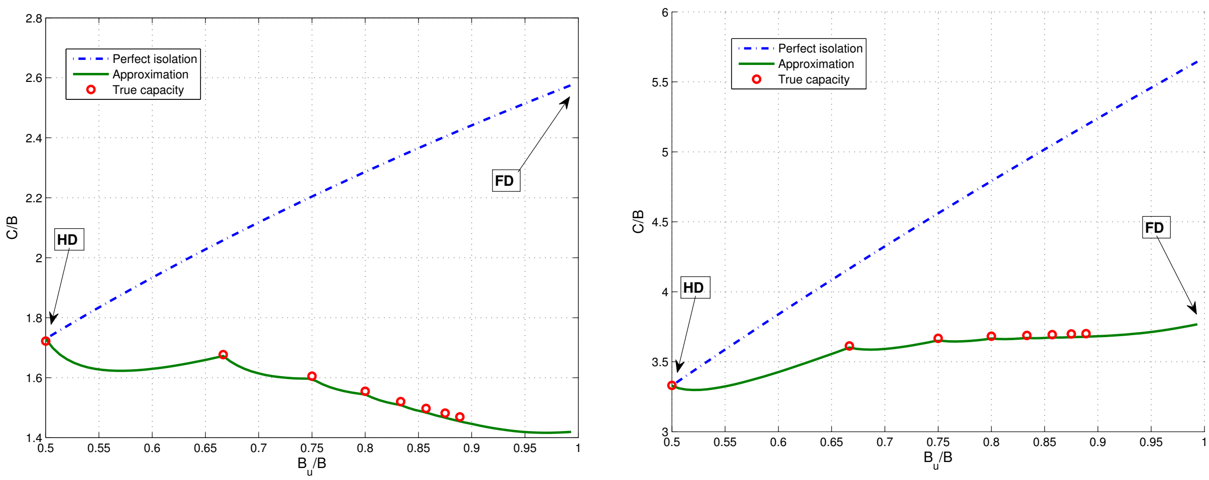

## Acknowledgment

This work was partially funded by the Spanish Ministry of Economy and Competitiveness and the European Regional Development Fund (ERDF) under project COMPASS (TEC2013-47020-C2-1-R), by the Galician Regional Government and ERDF under projects "Consolidation of Research Units" (GRC2013/009), REdTEIC (R2014/037) and AtlantTIC.




(a) snr = 7 dB. (b) snr = 17 dB.

Fig. 2: Relay achievable rate normalized by total bandwidth $B$, $\mathrm{LG} = 0$ dB.




(a) snr = 7 dB. (b) snr = 17 dB.

Fig. 3: Relay achievable rate normalized by total bandwidth $B$, $\mathrm{LG} = 5$ dB.

# References


[1] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Transactions on Information Theory*, vol. 50, no. 12, pp. 3062–3080, Dec 2004.

[2] R. Lopez-Valcarce, E. Antonio-Rodriguez, C. Mosquera, and F. Perez-Gonzalez, "An adaptive feedback canceller for full-duplex relays based on spectrum shaping," *Selected Areas in Communications, IEEE Journal on*, vol. 30, no. 8, pp. 1566–1577, September 2012.

[3] T. Riihonen, S. Werner, and R. Wichman, "Comparison of full-duplex and half-duplex modes with a fixed amplify-and-forward relay," in *Wireless Communications and Networking Conference, 2009. WCNC 2009. IEEE*, April 2009, pp. 1–5.

[4] K. Yamamoto, K. Haneda, H. Murata, and S. Yoshida, "Optimal transmission scheduling for a hybrid of full- and half-duplex relaying," *IEEE Communications Letters*, vol. 15, no. 3, pp. 305–307, March 2011.

[5] T. Riihonen, S. Werner, and R. Wichman, "Hybrid full-duplex/half-duplex relaying with transmit power adaptation," *IEEE Transactions on Wireless Communications*, vol. 10, no. 9, pp. 3074–3085, September 2011.

[6] N. Shlezinger and R. Dabora, "On the Capacity of Narrowband PLC Channels," *Communications, IEEE Transactions on*, vol. 63, no. 4, pp. 1191–1201, April 2015.

[7] L. Brandenburg and A. Wyner, "Capacity of the Gaussian channel with memory: The multivariate case," *Bell System Technical Journal, The*, vol. 53, no. 5, pp. 745–778, May 1974.