

Random Access Diversity through Beam Oversampling

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Abstract—The detection of random-access transmissions from a large population of uncoordinated terminals is a fundamental challenge in space-based monitoring of systems such as Automatic Identification System (AIS) or Automatic Dependent Surveillance-Broadcast (ADS-B). In these scenarios, a satellite receiver must reliably detect short messages. The spatial selectivity is ultimately limited by the on-board antenna size. In this paper, *beam oversampling* is proposed as a structured strategy to increase the probability of message detection by a multibeam satellite without relaxing the antenna constraint. Rather than relying on a single tessellation of the coverage area, the proposed scheme superposes multiple shifted copies of a baseline beam layout, so that every point on the ground is simultaneously illuminated by K beams. We show that the shift vectors can be chosen as the coset representatives of an index- K sub-lattice of the baseline lattice, a construction that guarantees spatially uniform detection probability for all user locations. The probability of detection is derived via the Probability of the Inclusion-Exclusion (PIE) principle, with numerical results confirming the associated detection gain at the cost of multiplying the number of beams to process by a factor K : up to several-fold gains in detection probability can be achieved for moderate traffic levels and modest K .

Index Terms—Poisson, random access, AIS, ADS-B, multi-beam, lattice.

I. INTRODUCTION

Random access protocols are widely adopted in large-scale wireless networks owing to their simplicity and decentralized nature: nodes transmit without prior coordination, making them well suited to scenarios where the number of active devices is unknown or highly variable. The fundamental limitation of uncoordinated random access is the occurrence of message collisions, which arise whenever two or more devices transmit simultaneously over the same channel resource, so that the resulting interference may prevent successful decoding.

Colliding messages can be recovered in some cases, especially when protocols are designed to this effect, with coded random access (CRA) as the main exponent [1]: each node transmits multiple replicas of its message in such a way that interference at the receiver can be canceled by means of iterative interference cancellation. On the other hand, many legacy

random access protocols lack this structure, and reduction of collisions must be pursued by reducing the probability that two transmissions interfere in the first place. Spatial selectivity is a natural tool in this direction: by restricting reception to a limited angular or geographic region through directional beamforming or spatially selective beam patterns, the number of devices whose signals are simultaneously captured decreases, and the collision probability is correspondingly reduced [2]. However, spatial selectivity is constrained by the receiving platform, as the antenna size determines the beamwidth [3].

The detection of maritime and aeronautical positioning signals from low-Earth orbit (LEO) satellites represents a compelling application domain for advanced beamforming strategies in random access networks. The Automatic Identification System (AIS) for maritime vessels [4], and the Automatic Dependent Surveillance-Broadcast (ADS-B) system [5], which broadcasts the position and identity of aircraft, both operate as open-loop random access protocols: transmitters emit short messages without a true coordination, and no acknowledgment or retransmission mechanism is available. While ground-based receiver networks provide adequate coverage in coastal and continental regions, vast oceanic and polar areas remain beyond terrestrial infrastructure reach. Satellite-based reception introduces a fundamental scalability challenge: a single LEO satellite sweeps a large footprint on the Earth's surface, potentially capturing thousands of simultaneous transmitters within its field of view. Under such conditions, the message collision rate becomes the dominant performance bottleneck, as the probability that two or more vessels or aircraft transmit simultaneously grows rapidly with traffic density. Spatially selective beamforming offers a principled mitigation strategy: by subdividing the satellite footprint into a mosaic of narrow directional beams, each beam observes a smaller geographic cell containing fewer co-channel transmitters, thereby reducing the collision probability and increasing the probability of successfully decoding individual messages.

In this work, we analyze how the geometry of the beam layout, in particular the use of multiple shifted replicas of a baseline tessellation, can provide a fruitful diversity through the smart use of beam overlapping, oriented to improve the detection performance. Some related and inspiring work include Li et al. [6] and Artiga et al. [7], that investigate, among

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other aspects, how increasing the number of beams enhances the decoding probability in space-based ADS-B systems with multibeam arrays. For space-based AIS reception, in [8] the beam distribution is heavily oversampled and various sidelobe levels are explored using the Chebyshev distribution, which enables control over the constant sidelobe level. As a result, interfering messages may experience different array gains depending on their angular positions. This variation introduces diversity to the single-channel detector, potentially improving the probability of successful detection in some instances.

Next section details our system model, prior to the description and analysis of the proposed beam oversampling scheme in Section III. Finally, some numerical results are presented for illustration purposes before sharing the conclusions.

II. SYSTEM MODEL

User devices¹ are assumed to be distributed across the coverage area according to a homogeneous Poisson point process (PPP) Ψ on $\mathbb{R}^2$, with spatial density ρ (devices per unit area). Each device transmits messages independently, with probability p_{tx} per unit time interval. Under these assumptions, the set of devices that are simultaneously active during any given observation interval, hereafter referred to as a *slot* for brevity, without implying a slotted access protocol, forms a thinned PPP $\Phi \subset \Psi$, which is itself a homogeneous PPP with density [9]

$$\lambda = \rho \cdot p_{tx} \quad [\text{packets per unit area per slot}]. \quad (1)$$

The parameter λ therefore represents the average number of messages or packets² simultaneously present in the channel per unit area, and constitutes the sole traffic parameter in the subsequent analysis. This abstraction captures the essential statistical behavior of uncoordinated random-access transmissions: the spatial locations of active emitters are unknown to the receiver and mutually independent, and the packet generation process is memoryless.

The baseline beam layout follows a tessellation of hexagons with apothem r_0 , as approximation to circular beams of the same radius, so every user is always within the main lobe of at least one beam (Fig. 1). The hexagons are arranged on a regular triangular lattice Λ with inter-element spacing $d_0 = 2r_0$. In the figures and simulations, r_0 will be normalized to 1. The multibeam pattern is generated by a digital array operating in beam processing mode [10]. We will assume ideal identical beams, with no side lobes³.

We want to exploit the diversity generated by beam overlapping towards the improvement of the probability of detection metric P_d . For simplicity, we will consider that collisions cannot be solved, and the involved packets are lost.

¹For example, AIS or ADS-B transponders on-board ships or aircraft, respectively.

²Throughout this paper, the terms packet and message are used interchangeably to refer to the basic unit of data transmission.

³Beamwidth can be assumed constant across steering directions if we neglect the widening of the beam shape away from boresight. Alternatively, operation with (u, v) coordinates yields identical beams [7].

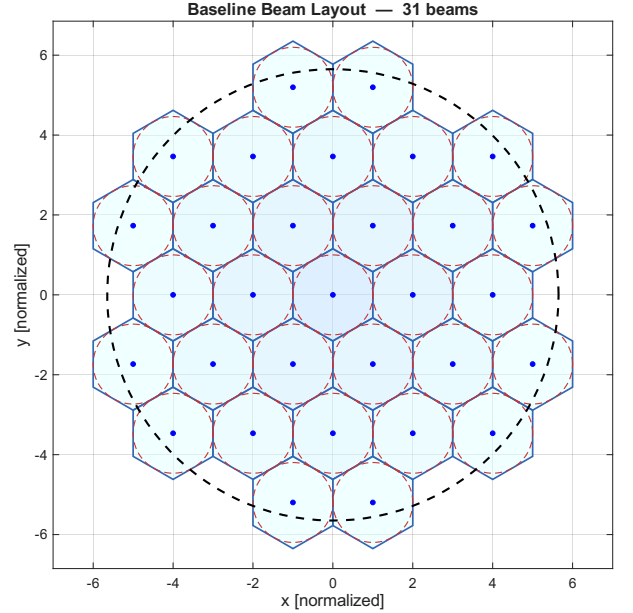


Fig. 1: Baseline layout.

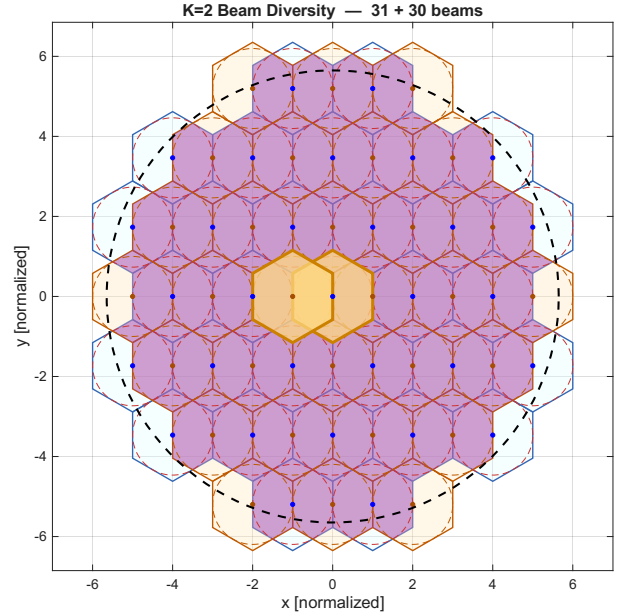


Fig. 2: Diversity layout, $K = 2$.

By increasing the beam density by a factor of K , every point in the surveillance area is simultaneously observed by K identical beams, in what will be referred to as **beam oversampling**. The density of centroids of the tessellation increases by the overlapping factor K . The replication of the baseline tessellation introduces spatial diversity: even if a collision prevents detection in one beam, the same emitter is also observed by the remaining $K - 1$ beams, and chances to find at least one observation free of collisions increase (see Fig. 2 for $K = 2$).

III. BEAM OVERSAMPLING

The minimum beam size is usually constrained by the antenna size of the receiver (for example, a satellite on the sky), so the maximum spatial selectivity is limited, and some other mechanisms must be explored to improve the packet detection statistics. Thus, it is interesting to evaluate the potential of increasing the number of beams for a given beamwidth. Furthermore, in practical settings, as user distribution can be geography dependent, a varying beam density can be also of interest [10].

To achieve K -fold spatial diversity, the baseline lattice Λ is replicated K times, each copy being rigidly translated by the corresponding shift vector $\mathbf{s}_k \in \mathbb{R}^2$, $k = 1, \dots, K$. Denoting the basis vectors of Λ by $\mathbf{e}_1$ and $\mathbf{e}_2$, the k -th shifted lattice (coset) is [11]

$$\Lambda_k = \Lambda + \mathbf{s}_k = \{m\mathbf{e}_1 + n\mathbf{e}_2 + \mathbf{s}_k : m, n \in \mathbb{Z}\}, \quad k = 1, \dots, K. \quad (2)$$

The K shift vectors $\{\mathbf{s}_k\}$ are chosen as K distinct coset representatives of Λ , so that the K shifted lattices $\Lambda_1, \dots, \Lambda_K$ are mutually distinct. Since each Λ_k independently tiles the plane, every point in $\mathbb{R}^2$ is covered by exactly one beam from each copy, yielding uniform K -fold coverage.

A tagged emitter located at position $\mathbf{p} \in \mathbb{R}^2$ is covered by the hexagonal beams centered at the nearest points of Λ_k , $k = 1, \dots, K$ to $\mathbf{p}$. Since the k -th copy is a shifted version of Λ , the nearest centroid in Λ_k to $\mathbf{p}$ is uniquely determined by the fractional part of $\mathbf{p}$ relative to Λ_k . Crucially, the relative displacement between the K covering centroids, one per copy, equals the fixed vector $\mathbf{s}_j - \mathbf{s}_k$ for every pair (j, k) , regardless of where $\mathbf{p}$ lies. More precisely, if $\mathbf{c}_k(\mathbf{p})$ denotes the centroid in Λ_k covering $\mathbf{p}$, then

$$\mathbf{c}_j(\mathbf{p}) - \mathbf{c}_k(\mathbf{p}) = \mathbf{s}_j - \mathbf{s}_k, \quad \forall \mathbf{p} \in \mathbb{R}^2, \quad \forall j, k. \quad (3)$$

Therefore, the union of the K beams covering $\mathbf{p}$ is a fixed geometric shape, identical for all user locations, with area $A_{\cup}(\mathbf{s}_1, \dots, \mathbf{s}_K)$. As a consequence, the detection probability P_d is *spatially uniform*: it takes the same value for every user in the plane under a homogeneous traffic load. Note that we are not considering edge effects, since at the boundary of the coverage area the stationarity assumption is violated.

A. Detection Probability Under Beam Oversampling

The detection of a packet from a transmitter at $\mathbf{p}$ requires its detection by at least one of the K covering beams, which in our model requires the absence of collisions in at least one beam. As we will see, the probability of detection P_d will be uniform across the coverage area, so we can consider $\mathbf{p}$ to be at the origin without loss of generality. This representative transmitter constitutes the “typical user” of the system. By Slivnyak’s theorem [12], conditioning on the presence of a point of Φ at the origin leaves the remainder of the process distributed as an independent PPP with the same density λ , so that the detection probability can be evaluated directly in terms of the full traffic intensity without correction. We have from the Poisson statistics that, in the absence of diversity

($K = 1$), $P_d = e^{-G}$, with $G = \lambda\pi r_0^2$ the mean traffic within the baseline main lobe.

For the diversity case, with overlapping, and neglecting again the interference from the sidelobes, we need to estimate the collision-free probability in at least one of the overlapping beams. The event that the k -th beam is empty of transmissions is labeled as E_k . The probability of successful decoding is the probability of the union of the K events:

$$P_d(K) = P\left(\bigcup_{k=1}^K E_k\right). \quad (4)$$

This expression is exact under the model assumptions: ideal beams, no sidelobes, and a collision rule in which any additional active transmitter inside the same beam footprint prevents successful detection. The difficulty is that the events $E_1, \dots, E_K$ are not independent. Since the beams overlap, the same active packet may fall inside more than one of the K footprints. Therefore, the union probability needs to correct for the repeated counting of overlaps among events. By the Principle of Inclusion-Exclusion (PIE) [12], this probability is expressed as

$$P_d(K) = \sum_{m=1}^K (-1)^{m-1} S_m, \quad (5)$$

where S_m is the total contribution of all groups of m beams that are simultaneously collision-free:

$$S_m = \sum_{1 \leq i_1 < \dots < i_m \leq K} P\left(\bigcap_{n=1}^m E_{i_n}\right) \quad (6)$$

and $\{i_m\}$ the indexes of beams covering the point of interest. The alternating signs in the PIE expansion (5) ensure that overlaps are counted only once.

Note that $P(\bigcap_{n=1}^m E_{i_n})$ is the probability of successful reception of a user located in the intersection of beams $\{1, \dots, m\}$, i.e., the probability of not finding colliding packets in the union of beams $\{1, \dots, m\}$:

$$P\left(\bigcap_{n=1}^m E_{i_n}\right) = \exp(-\lambda \cdot A_{\cup}(\{i_1, \dots, i_m\})) \quad (7)$$

where $A_{\cup}(\{i_1, \dots, i_m\})$ is the area of the union of the m intersecting beams, which in general must be computed numerically. $P_d(K)$ will be a function of the traffic G and the replication strategy, so it is more correctly expressed as

$$P_d(G; \mathbf{s}_1, \dots, \mathbf{s}_K) \triangleq P_d(K), \quad (8)$$

as the detection probability is governed by the interplay between beam overlap geometry and random spatial traffic.

B. Replication Strategies

The choice of shift vectors $\{\mathbf{s}_k\}$ constitutes the *replication strategy*. Different strategies place the K coset representatives at different locations relative to the underlying lattice, leading to different inter-centroid distances and thus different intersection areas and values of $P_d(K)$. Three natural strategies are described below.

a) *One-Dimensional (1D) Grid Strategy*: For arbitrary K , the shifts can be placed along the primitive vector $\mathbf{e}_1$ at equally spaced intervals:

$$\mathbf{s}_k = \frac{k-1}{K} \mathbf{e}_1, \quad k = 1, \dots, K. \quad (9)$$

This strategy is applicable for any integer $K \geq 1$. The K centroids are collinear, and adjacent centroids are separated by the constant distance $2r_0/K$. This collinear arrangement produces larger pairwise intersection areas compared to the following 2D strategy.

b) *Rectangular Grid Strategy*: A more isotropic arrangement for arbitrary K places the shifts on a two-dimensional sub-grid. Writing $K = p \times q$ with $p, q \in \mathbb{Z}^+$, the shifts are

$$\mathbf{s}_{i,j} = \frac{i}{p} \mathbf{e}_1 + \frac{j}{q} \mathbf{e}_2, \quad i = 0, \dots, p-1, \quad j = 0, \dots, q-1, \quad (10)$$

This distributes the K centroids more evenly in two dimensions, reducing the maximum pairwise intersection area compared to the 1D strategy, which is a particular case.

For illustration purposes, Fig. 3 shows the basic tessellation and shifting vectors for $K = 4$.

c) *Eisenstein Strategy*: For certain values of K of the form $K = a^2 + ab + b^2$ with non-negative integers a, b , the so-called *Eisenstein integers*, the K shift vectors can be chosen as $\mathbf{s}_{ij} = i \mathbf{w}_1 + j \mathbf{w}_2$ for those integer pairs (i, j) such that $\mathbf{s}_{ij} \in \mathcal{V}(\Lambda)$, and

$$\mathbf{w}_1 = \frac{a\mathbf{e}_1 + b\mathbf{e}_2}{K}, \quad \mathbf{w}_2 = \frac{-b\mathbf{e}_1 + (a+b)\mathbf{e}_2}{K}. \quad (11)$$

This construction maximizes the minimum inter-centroid distance among all strategies for the same K , since the K coset representatives form the densest possible placement of K points in the plane under the given lattice structure. Valid values include $K \in \{1, 3, 4, 7, 9, 12, 13, \dots\}$. For these values, the K shifts can be arranged on a triangular sub-grid which maximizes the minimum inter-centroid distance and thus tends to reduce overlap area.

These three techniques can be embedded into a general parameterization by an integer matrix M_{HNF} in the form of the Hermite Normal Form (HNF) [13], a 2×2 integer upper triangular matrix, with off-diagonal elements smaller than pivot, such that $|\det M_{\text{HNF}}| = K$:

$$M_{\text{HNF}} = \begin{pmatrix} K/d & c \\ 0 & d \end{pmatrix}, \quad d \mid K, \quad 0 \leq c < K/d, \quad (12)$$

where d ranges over all positive divisors of K and c is an integer offset. The K coset representatives are then

$$\mathbf{s}_{i,j} = \frac{i}{K/d} \mathbf{e}_1 + \frac{j}{d} \mathbf{e}_2, \quad i = 0, \dots, K/d-1, \quad j = 0, \dots, d-1. \quad (13)$$

As a remark, note that in the Eisenstein case, the natural generator matrix is

$$M_{\text{Eis}} = \begin{pmatrix} a & -b \\ b & a+b \end{pmatrix}, \quad \det M_{\text{Eis}} = K, \quad (14)$$

which is not itself HNF; its HNF equivalent $H = M_{\text{Eis}} U$, for some unimodular matrix $U \in GL_2(\mathbb{Z})$, has $c \neq 0$ in (12).

Some differences in terms of $P_d(K)$ can be expected for certain K values among the different replication schemes. We will detail in the simulations which one is used at each case.

C. Asymptotic Behavior

In the dense-packing limit, the discrete oversampling effectively approximates a continuous scan over the location of the reference user. The probability that an ideal circular beam of radius r_0 covers the user of interest and no other user reduces to a classical geometric probability problem, one for which no closed-form solution is known in the general case.

Specifically, this probability is equivalent to the probability that the disk $B_{r_0}(0)$ of radius r_0 , centered at the origin, is not covered by the Boolean model Z defined as follows. Let $\eta = x_1, x_2, \dots$ be a homogeneous spatial Poisson point process of intensity λ on $\mathbb{R}^2$. The associated Boolean model⁴ is the stationary random closed set Z formed by the union of disks centered at each point of η [9]:

$$Z \triangleq \bigcup_n B_{r_0}(x_n) \quad (15)$$

where each grain $B_{r_0}(x_n)$ is a disk of radius r_0 centered at the germ $x_n \in \eta$. If the intersection of the coverage process Z with $B_{r_0}(0)$ is contained within $B_{r_0}(0)$, this means that there will be at least one location at a distance higher than r_0 from all points in η , and lower than r_0 to the origin.

This non-coverage probability corresponds to the limit of expression (5), for which no solution is known. This problem or, equivalently, its coverage counterpart, has only been partially addressed to date, and proven quite elusive, with early attempts of circular coverage situations developed, among other applications, for bombing problems [14]. Asymptotic results for large traffic values are presented in the seminal book [15], and recent approaches include works such as [16]. For large load $G = \lambda \pi r_0^2$, the vacancy probability is [17]

$$Pr(Z \cap B_{r_0}(0) \subset B_{r_0}(0)) \approx 1 - e^{-G^2 e^{-G}}. \quad (16)$$

Essentially, as G grows, the number of holes or *vacancies* behaves as a Poisson random variable with expectation $\lambda_{as} = G^2 e^{-G}$. The expression in (16) follows then from $1 - e^{-\lambda_{as}}$, corresponding to the probability of having one or more occurrences of the Poisson random variable. In the next section, we will use a Monte Carlo approximation for $P_d(\infty)$ as upper performance limit for any beam oversampling scheme.

IV. NUMERICAL RESULTS

We have evaluated the probability of detection for different oversampling factors K applied to the baseline layout in Fig. 1. Traffic is modeled as a homogeneous PPP with $G = \lambda \pi$ packets per beam during one time slot. The replication strategies are chosen as one-dimensional for $K = 2$, rectangular

⁴When the underlying point process is Poisson, the model is also referred to as a Boolean-Poisson model.

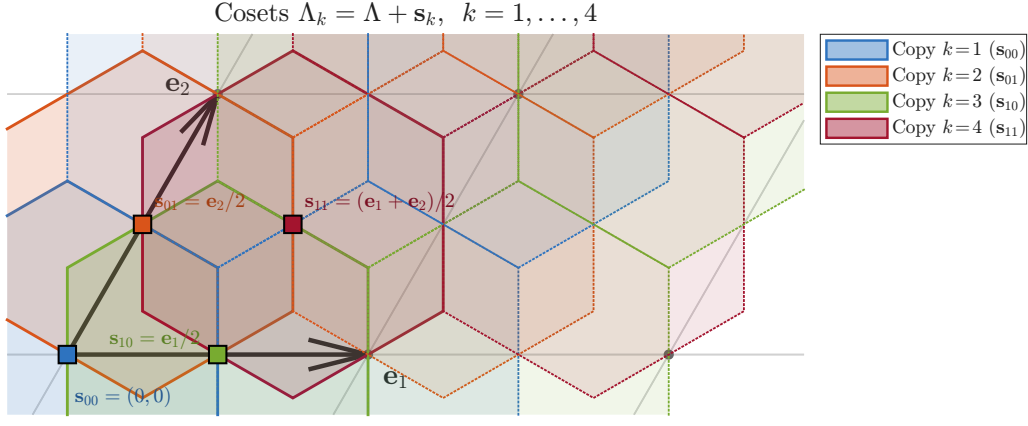


Fig. 3: Replication for $K = 4$.

for $K = \{4, 6, 9\}$, and Eisenstein for $K = 3$. Note that for $K = \{4, 9\}$, the two latter are equivalent. Results are displayed in Fig. 4, along with the limit for the beam density going to infinity, which was obtained by Monte Carlo simulation, by exploiting the equivalence with the coverage problem mentioned above. The asymptotic analytical expression (16) is also shown for reference. Monte Carlo simulations (100 realizations) were also performed for three values of K , and the results closely match the analytical predictions⁵, as shown in Fig. 5. As expected, a notable improvement in $P_d(K)$ takes place for modest values of K , which in practice can boost the performance in low-to-moderate traffic regimes if $P_d(1)$ does not meet the required quality. Substantial, up to several-fold, gains in detection probability can be achieved for moderate traffic levels. Thus, for $G = 2$ packets/beam, P_d increases from 0.1 at $K = 1$ to 0.27 at $K = 3$, and 0.4 at $K = 6$. From a practical standpoint, a spatially adaptive beam density, tuned to the local traffic load, is of interest to optimize the global number of processed beams [7].

V. CONCLUSIONS

This paper has presented a mathematical framework for K -fold beam diversity through the superposition of shifted copies of a baseline hexagonal beam layout. The proposed beam over-sampling strategy guarantees spatially uniform probability of detection for all user locations, computed through the Principle of Inclusion-Exclusion. The processing of all the overlapping beams improves the probability of detection of packets in the presence of random collisions. With this, limited spatial selectivity can be overcome in multibeam arrangements for the space-based monitoring of maritime and aeronautical signaling systems. Further extensions must consider non-ideal beam patterns with smooth gain roll-off, and the replacement of the binary coverage criterion by a Signal-to-Interference-and-Noise Ratio (SINR) threshold condition, for a more faithful description of the actual performance. Validation under realistic traffic conditions, potentially departing from the ideal Poisson

⁵Locations near the edge, for which assumptions do not hold, are not included.

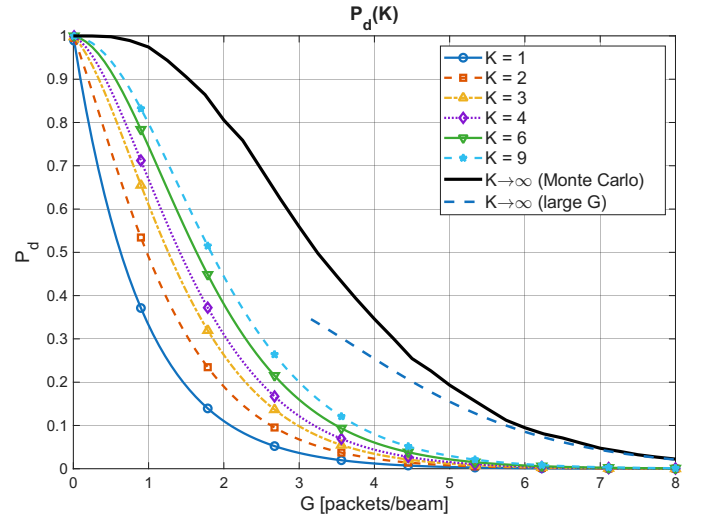


Fig. 4: Detection probability versus traffic intensity.

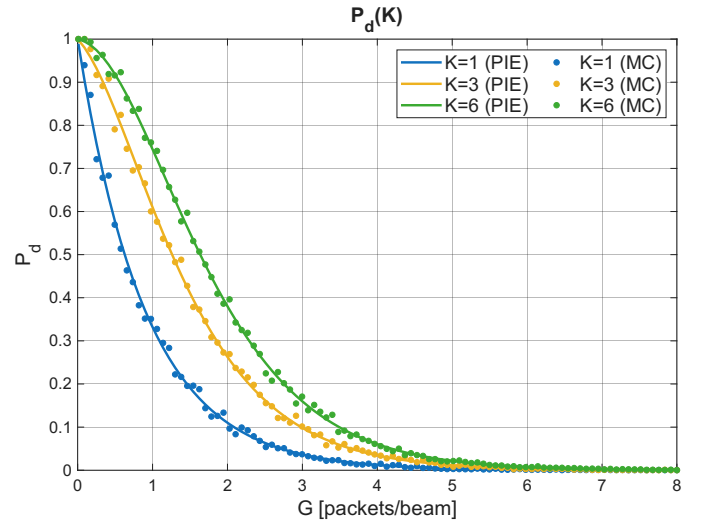


Fig. 5: Detection probability versus traffic intensity. Analytical results (PIE) and Monte Carlo simulations (MC).

model, is also desirable for both maritime and aeronautical applications.

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